Asymptotic phase shifts and Levinson theorem for 2D potentials with inverse square singularities

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Abstract

The Levinson theorem for two–dimensional scattering is generalized for potentials with inverse square singularities. By this theorem, the number of bound states in a given m–th partial wave is related to the phase shift and the singularity strength of the potential. For the m–wave phase shift the asymptotic behaviour is calculated for short wavelengths.

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I. INTRODUCTION

The Levinson theorem sets up a relation between the number of bound (b) states $N_l^{\rm b}$ in a given l-th partial wave and the phase shift $\delta_l(k)$. The theorem was proved for three-dimensional (3D) central potentials $V(|\boldsymbol{r}|)$, see the review [1, 2]. Levinson's relation for the l–wave phase shift gives $\delta_l(0) - \delta_l(\infty) = \pi N_l^{\rm b}$. If the half–bound (hb) state occurs for the s-wave type ($l = 0$), this is modified to $\delta_0(0) - \delta_0(\infty) = \pi(N_0^{\rm b} + \frac{1}{2})$. The Levinson theorem is one of the most beautiful results of scattering theory; it was a subject of studies by many authors.

The Levinson theorem in 3D has been discussed for noncentral potentials [2, 3, 4], singular potentials [5], energy–dependent potentials [6], nonlocal interactions [7], Dirac particles [8, 9], systems with coupling [10], multichannel scattering [11, 12], multiparticle single–channel scattering [13], and in the inverse scattering theory, even with singular potentials [14, 15, 16].

Recently, the Levinson theorem was established for lower–dimensional systems, which play an important role in modern physics of condensed matter and in field theories. The 1D Levinson theorem was validated for the Schrödinger equation [17], the Schrödinger equation with a nonlocal interaction [18], the Klein–Gordon equation [19], and the Dirac equation [20], even in the presence of solitons [21]. The Levinson theorem was implemented in the (1+1) gauge theory to calculate the fractional and integer fermion numbers [22].

Let us consider 2D systems. The 2D Levinson theorem was established for different models, too: for the Schrödinger equation [23, 24], the Klein–Gordon equation [25], and the Dirac equation [26]. Moreover there exists an extension of the Levinson theorem for the Schrödinger equation in D dimensions. There are several methods for studying the lower-dimensional Levinson theorem: the Jost function method [27], the Green function method [20, 23, 26, 28], and the Sturm–Liouville theorem [17, 18, 19, 24, 25, 29]. Levinson's relation for the partial wave phase has the usual form, as for the 3D case; but the half–bound state for the p–wave ($l = 1$) contributes exactly like the bound state and gives an additional π to Levinson's relation [28]. Let us remind that a half–bound state is the zero–energy solution for the case when the eigenfunction is finite, but does not decay fast enough at infinity to be square integrable. In the 2D case a possible s–wave half–bound state does not contribute at all to Levinson's relation, but only the p–wave half–bound state. An experimental justification of the Levinson theorem in the 2D case was made in Refs. [30, 31] for the 2D plasma. All mentioned papers, which discuss the 2D version of the Levinson theorem, consider potentials which are less singular than $|\boldsymbol{r}|^{-2}$. This is a standard assumption, which results in the above mentioned form of the Levinson theorem.

At present singular potentials become an object of interest. Singular potentials naturally appear in singular inverse problems, i.e. in a supersymmetric approach to the inverse scattering in 3D, when bound states are removed from the regular potential [14, 15, 16]. The short distance behaviour of the singular potential is defined by the inverse square asymptotics at the origin $V(r) \sim \beta_0/r^2$; therefore the resulting effective potential for the partial wave U_l (*partial potential*) in the 3D case has the asymptotic form

$$U_l(r) = V(r) + \frac{l(l+1)}{r^2} \underset{r\to 0}{\sim} \frac{\nu(\nu+1)}{r^2},$$

with the *singularity strength* $\nu = \sqrt{(l+1/2)^2+\beta_0} - 1/2 \neq l$. One can see that the singular potential acts as a correction to the centrifugal barrier $l(l+1)/r^2$. The scattering problem for such potentials with an inverse square singularity was solved firstly by Swan, who has generalized the Levinson theorem for singular potentials in the 3D case [5]. It reads:

$$\delta_l(0) - \delta_l(\infty) = \pi \cdot \left(N_l^{\mathrm{b}} + \frac{\nu - l}{2} \right). \tag{1}$$

In addition to the general importance for the scattering theory, the generalized Levinson theorem (1) is useful for the inverse scattering theory, because it gives a possibility to determine the parameter of the singular core of the potential from the scattering data.

In the present paper we establish the 2D analogue of the generalized Levinson theorem (1). Singular potentials appear in different 2D systems: in the (2+1)-dimensional O(3)–models like $3D-SU(N_f)$ skyrmions in N_f–flavor meson fields [32]; in the $2D-O(3)$ spin textures as charged quasi–particles in ferromagnetic quantum Hall systems [33]; in different models of 2D magnets as an effective potential of soliton (vortex)–magnon interaction [34, 35, 36, 37].

The paper is organized as follows. In Sec. II we formulate the scattering problem in the 2D case. We discuss the possible supersymmetric nature of singular potentials. The scattering problem is solved for the simplest example of a singular potential, i.e. for the centrifugal model, in Sec. III. A simple qualitative picture of the scattering problem is discussed in Sec. IV. In this section we calculate the phase shift in the short–wavelength limit. The generalized Levinson theorem is proved in Sec. V. A discussion and concluding remarks are presented in Sec. VI.

II. SCATTERING IN TWO DIMENSIONS: NOTATIONS, PARTIAL WAVE METHOD, SINGULAR POTENTIALS

Let us consider the Schrödinger–like equation in two dimensions:

$$-\nabla^2\Psi + V(\boldsymbol{r})\Psi = i\partial_t\Psi. \tag{2}$$

For the central (axially symmetric) potentials, $V(\boldsymbol{r}) = V(\rho)$, we apply the standard partial wave expansion, using the *ansatz*

$$\Psi(\boldsymbol{r},t) = \sum_{m=-\infty}^{\infty} \psi_m^{\mathcal{E}}(\rho)\cdot\exp(im\chi - i\mathcal{E}t) \ , \tag{3}$$

where (ρ,χ) are the polar coordinates in two spatial dimensions, $\{m,\mathcal{E}\}$ the complete set of eigennumbers, $\mathcal{E}$ and m the energy and the azimuthal quantum number, respectively. Each partial wave $\psi_m^{\mathcal{E}}$ is an eigenfunction of the spectral problem

$$H\psi_m^{\mathcal{E}}(\rho) = \mathcal{E}\psi_m^{\mathcal{E}}(\rho) \tag{4a}$$

for the 2D radial Schrödinger operator $H = -\nabla_\rho^2 + U_m(\rho)$ with the partial potential

$$U_m(\rho) = V(\rho) + \frac{m^2}{\rho^2} \ . \tag{4b}$$

Let us formulate the scattering problem. The continuum spectrum exists for $\mathcal{E} > 0$. Note that the eigenfunctions for the free particle, $V(\rho) = 0$, have the form

$$\psi_m^{\text{free}}(\rho) \propto J_m(k\rho), \qquad k = \sqrt{\mathcal{E}} > 0 \ , \tag{5}$$

where k is a "radial wave number", and J_m is a Bessel function. The free eigenfunctions like ψ_m^{free} play the role of partial cylinder waves of the plane wave

$$\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\mathcal{E}t) = \sum_{m=-\infty}^{\infty} i^m J_m(k\rho)e^{im\chi - i\mathcal{E}t}. \tag{6}$$

The behaviour of the eigenfunctions in the potential $V(\rho)$ can be analyzed at large distances from the origin, $\rho \gg R$, where R is a typical range of the potential $V(\rho)$. In view of the asymptotic behaviour $U_m(\rho) \sim m^2/\rho^2$, which is valid for fast decreasing potentials $V(\rho)$, in the leading approximation in $1/\rho$ we have the usual result

$$\psi_m^{\mathcal{E}} \propto J_{|m|}(k\rho) + \sigma_m(k)Y_{|m|}(k\rho) \ , \tag{7a}$$

where Y_m is a Neumann function. The quantity $\sigma_m(k)$ stems from the scattering; it can be interpreted as the scattering amplitude. In the limiting case $k\rho \gg |m|$ it is convenient to consider the asymptotic form of Eq. (7a),

$$\psi_m^{\mathcal{E}} \propto \frac{1}{\sqrt{\rho}} \cos\left(k\rho - \frac{|m|\pi}{2} - \frac{\pi}{4} + \delta_m(k)\right), \tag{7b}$$

where the scattering phase, or the phase shift $\delta_m(k) = -\arctan\sigma_m(k)$. The phase shift contains all informations about the scattering process. In particular, we give the general solution of the scattering problem for the plane wave (6). With Eqs. (3) and (7a), the asymptotic solution of the Schrödinger–like equation (2) for $\rho \gg R$ can be written

$$\begin{aligned}\Psi(\boldsymbol{r},t) = \sum_{m=-\infty}^{\infty} C_m \left(J_{|m|}(k\rho) + \sigma_m(k) Y_{|m|}(k\rho)\right) \\ \times \exp(im\chi - i\mathcal{E}t)\ ,\end{aligned} \tag{8}$$

where C_m are constants. To solve the scattering problem for the plane wave let us choose the constants C_m by comparing Eq. (8) with the expansion (5) for the free motion. Using the asymptotic forms for the cylinder functions in the region $\rho \gg 1/k$, we obtain

$$\begin{aligned}\Psi(\boldsymbol{r},t) &= e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\mathcal{E}t} + \mathcal{F}(\chi)\frac{e^{ik\rho-i\mathcal{E}t}}{\sqrt{\rho}}, \\ \mathcal{F}(\chi) &= \frac{\exp(-i\pi/4)}{\sqrt{2\pi k}} \cdot \sum_{m=-\infty}^{\infty} \left(e^{2i\delta_m} - 1\right) \cdot e^{im\chi}.\end{aligned} \tag{9}$$

The total scattering cross section is given by the expression

$$\varrho = \int_0^{2\pi} |\mathcal{F}|^2 d\chi = \sum_{m=-\infty}^{\infty} \varrho_m\ ,$$

where $\varrho_m = (4/k)\sin^2\delta_m$ are the partial scattering cross sections.

For regular 2D potentials $V(\rho)$, the 2D analogue of the Levinson theorem has the form [23, 24, 28]

$$\delta_m(0) - \delta_m(\infty) = \pi \cdot \left(N_m^{\mathrm{b}} + N_m^{\mathrm{hb}} \cdot \delta_{|m|,1}\right). \tag{10}$$

Here the potential $V(\rho)$ satisfies the asymptotic conditions

$$\lim_{\rho=0} \rho^2 V(\rho) \ = \ 0, \tag{11a}$$

$$\lim_{\rho=\infty} \rho^2 V(\rho) \ = \ 0, \tag{11b}$$

which provide a regular behaviour at the origin, and fast decaying at infinity.

Now we switch to the singular potentials, having in mind to reestablish the Levinson theorem.

A. Potentials with inverse square singularity

Let us consider potentials with inverse square singularity. At the origin, the potential has an asymptotics like $V(\rho) \sim \beta_0/\rho^2$; the corresponding partial potential (4b)

$$U_m(\rho) \underset{\rho\to 0}{\sim} \frac{\nu^2}{\rho^2}\,, \quad \text{with} \quad \nu = \sqrt{m^2+\beta_0} \neq m\,. \tag{12}$$

Singular potentials like (12) appear in various 2D non–linear field theories, e.g. for the scattering problem of linear excitations by topological solitons [32, 33, 34, 35, 36].

Moreover singular potentials naturally appear from regular ones under Darboux transformations [14, 15, 16, 38]. Let us recall the principle of Darboux (supersymmetric) transformations for the 2D case [35]. We suppose that the spectral problem (4a) has at least one bound state $\mathcal{E}_0 < 0$. Assuming that we start from the regular potential under conditions (11), then the eigenfunction $\psi_0 \equiv \psi_m^{\mathcal{E}_0}(\rho)$ may have the following asymptotic behaviour

$$\psi_0(\rho) \propto \begin{cases} \rho^{|m|}, & \text{when } \rho\to 0\,, \\ \rho^{-1/2}\cdot\exp\left(-\kappa\rho\right), & \text{when } \rho\to\infty\,, \end{cases} \tag{13}$$

where $\kappa = \sqrt{-\mathcal{E}_0} > 0$.

To explain the method we introduce the Hermitian–conjugate lowering and raising operators [35]

$$A = -\frac{d}{d\rho} + W(\rho), \qquad A^\dagger = \frac{d}{d\rho} + \frac{1}{\rho} + W(\rho), \tag{14}$$

where the superpotential

$$W(\rho) = \frac{d}{d\rho}\ln\psi_0 \tag{15}$$

is such that $A\psi_0 = 0$. By introducing these operators we can represent the Schrödinger operator H in the factorized form

$$H = A^\dagger A + \mathcal{E}_0\,, \tag{16}$$

the factorization energy $\mathcal{E}_0$ coincides with the energy of the bound state. Such a factorization makes it possible to reformulate the initial problem (16) in terms of the eigenfunction $\tilde{\psi}_m = A\psi_m$ of the spectral problem

$$\tilde{H} = AA^\dagger + \mathcal{E}_0 = -\nabla_\rho^2 + \tilde{U}_m(\rho)\,, \tag{17}$$

where the partial potential

$$\tilde{U}_m(\rho) = U_m(\rho) + \frac{1}{\rho^2} - 2\frac{d}{d\rho}W(\rho) . \tag{18}$$

Taking into account the conditions (13), one can derive the asymptotic behaviour of the partial potential $\tilde{U}_m$,

$$\tilde{U}_m(\rho) \sim \begin{cases} \dfrac{\nu^2}{\rho^2} \quad \text{with } \nu = |m| - 1, & \text{when } \rho \to 0 , \\ \dfrac{m^2}{\rho^2}, & \text{when } \rho \to \infty . \end{cases} \tag{19}$$

We see that the eigenspectrum of the new spectral problem (17) does not contain the bound state ψ_0. The resulting potential has a singularity; in fact, the partial potential $\tilde{U}_m(\rho)$ corresponds to the particle potential $V(\rho) = \beta/\rho^2$ with the parameter $\beta = 1 - 2|m|$. After a series of n transformations like (17), we remove n bound states from the spectrum, which results in $\tilde{U}_m \sim \nu^2/\rho^2$, with $\nu = |m| - n$.

B. Potentials with inverse square tail

Let us discuss potentials with an inverse square tail, when far from the origin the potential $V(\rho) \sim \beta_\infty/\rho^2$; the corresponding partial potential

$$U_m(\rho) \underset{\rho\to\infty}{\sim} \frac{\mu^2}{\rho^2} , \quad \text{with} \quad \mu = \sqrt{m^2 + \beta_\infty} \neq m . \tag{20}$$

Potentials like (20) are of interest in field theories: in the (2+1) nonlinear σ–model of the $\boldsymbol{n}$–field [32, 35], in models of 2D easy–axis [36] and easy–plane ferromagnets in the cone state [37].

To study the scattering problem let us consider the asymptotic behaviour of the eigenfunctions. Obviously, at large distances $\rho \gg R$, where the scattering approximation is valid, one can use the partial wave expansion by the cylinder functions of the integer indexes only; then the eigenfunction $\psi_m^{\mathcal{E}}$ can be written as $J_{|m|} + \sigma_m Y_{|m|}$ with the asymptotic form (7b).

On the other hand, in the leading approximation in $1/k\rho$, the solution of the Schrödinger equation (4a) with the potential (20) can be written as

$$\begin{aligned} \psi_m^{\mathcal{E}}(\rho) &\propto J_{|\mu|}(k\rho) + \tilde{\sigma}_\mu(k) Y_{|\mu|}(k\rho) \\ &\propto \frac{1}{\sqrt{\rho}} \cos\left(k\rho - \frac{|\mu|\pi}{2} - \frac{\pi}{4} + \tilde{\delta}_\mu(k)\right) , \end{aligned} \tag{21}$$

where the index of the cylinder functions $\mu \neq m$, see Eq. (20).

The phase shift δ_m can be calculated from $\tilde{\delta}_\mu$ by comparing Eqs. (7b) and (21),

$$\delta_m(k) = \tilde{\delta}_\mu(k) + \frac{|m| - |\mu|}{2} \cdot \pi \ , \tag{22}$$

in accordance with the results of Refs. [24, 37]. Note that Levinson's relation has the same form for both phase shifts δ_m and $\tilde{\delta}_\mu$,

$$\delta_m(0) - \delta_m(\infty) = \tilde{\delta}_\mu(0) - \tilde{\delta}_\mu(\infty) \ .$$

III. SCATTERING PROBLEM FOR THE CENTRIFUGAL MODEL

For the analytical description of the scattering problem, let us consider the simplest model, which includes the main features of the problem, having both inverse square singularity and inverse square tail. The partial potential of this very simple *centrifugal model* [36] has the form

$$U_m^{\text{cf}}(\rho) = \begin{cases} \dfrac{\nu^2}{\rho^2} \ , & \text{when } \rho < R \ , \\ \dfrac{\mu^2}{\rho^2} \ , & \text{otherwise} \ , \end{cases} \tag{23}$$

with $\nu \neq m$, and $\mu \neq m$.

This model describes a quasi–free particle in each of the regions $\rho < R$ and $\rho > R$. The only effect of the interaction with the potential U_m^{cf} is a shift of the mode indices:

$$\psi_m^{\text{cf}}(r) \propto \begin{cases} J_{|\nu|}(k\rho) \ , & \text{when } \rho < R, \\ J_{|\mu|}(k\rho) + \tilde{\sigma}_\mu(k) Y_{|\mu|}(k\rho), & \text{otherwise} \ . \end{cases} \tag{24}$$

The usual matching condition for these solutions has the form

$$\left[\frac{\psi'}{\psi} \right]_R = 0 \ , \tag{25}$$

where $[\dots]_R \equiv (\dots)\big|_{R+0} - (\dots)\big|_{R-0}$, and the prime denotes $d/d\rho$. The calculations lead to the scattering phase shift in the form:

$$\begin{aligned} \delta_m^{\text{cf}}(k) &= \frac{|m| - |\mu|}{2} \cdot \pi - \arctan \tilde{\sigma}_\mu^{\text{cf}}(\varkappa \equiv kR) \ , \\ \tilde{\sigma}_\mu^{\text{cf}}(\varkappa) &= \frac{J'_{|\nu|}(\varkappa) \cdot J_{|\mu|}(\varkappa) - J'_{|\mu|}(\varkappa) \cdot J_{|\nu|}(\varkappa)}{J_{|\nu|}(\varkappa) \cdot Y'_{|\mu|}(\varkappa) - J'_{|\nu|}(\varkappa) \cdot Y_{|\mu|}(\varkappa)} \ . \end{aligned} \tag{26}$$

Using the asymptotic form of the cylinder functions, one can find the long– and short–wavelength behaviour of the phase shift (26),

$$\delta_m^{\mathrm{cf}}(k) \sim \begin{cases} \dfrac{|m|-|\mu|}{2}\cdot\pi + \mathcal{A}_m \cdot \left(\dfrac{kR}{2}\right)^{2|\mu|}, & kR \ll 1,\\ \dfrac{|m|-|\nu|}{2}\cdot\pi - \dfrac{\mu^2-\nu^2}{2kR}, & kR \gg 1, \end{cases} \tag{27}$$

where $\mathcal{A}_m = -\dfrac{\pi|\mu|}{(|\mu|)!^2}\cdot\dfrac{|\mu|+|\nu|}{|\mu|-|\nu|}$.

The Levinson theorem for the centrifugal model can be easily derived from Eq. (27):

$$\delta_m^{\mathrm{cf}}(0) - \delta_m^{\mathrm{cf}}(\infty) = \pi \cdot \frac{|\nu|-|\mu|}{2}\ . \tag{28}$$

IV. SCATTERING PROBLEM IN THE WKB APPROXIMATION

Now we discuss the general case where the partial potential has the asymptotic behaviour

$$U_m(\rho) \sim \begin{cases} \dfrac{\nu^2}{\rho^2}, & \text{when } \rho \to 0,\\ \dfrac{\mu^2}{\rho^2}, & \text{when } \rho \to \infty, \end{cases} \tag{29}$$

with $\nu \neq m$, and $\mu \neq m$.

The scattering problem can be treated analytically in the short–wavelength limit, $kR \gg 1$. It is natural to suppose that the WKB–approximation is valid for this case. We use the WKB–method in the form proposed earlier for the description of the scattering for isotropic 2D magnets [35]. We start from the effective 1D Schrödinger equation for the radial function $\psi_m(\rho) = u_m(\rho)/\sqrt{\rho}$, which yields

$$\begin{aligned} \left[-\frac{d^2}{d\rho^2} + \mathcal{U}_{\mathrm{eff}}(\rho)\right] u_m &= \mathcal{E} u_m\ ,\\ \mathcal{U}_{\mathrm{eff}}(\rho) &= V(\rho) + \frac{4m^2-1}{4\rho^2}\ . \end{aligned} \tag{30}$$

The WKB–solution of the Eq. (30), i.e. the 1D wave function u_m^{WKB}, leads to the following form of the partial wave

$$\psi_m^{WKB} = \frac{u_m^{WKB}}{\sqrt{\rho}} \propto \frac{1}{\sqrt{\rho\cdot\mathcal{P}(\rho)}}\cos\left(\chi_0 + \int_{\rho_0}^{\rho}\mathcal{P}(\rho')d\rho'\right), \tag{31}$$

where $\mathcal{P} = \sqrt{k^2 - \mathcal{U}_{\mathrm{eff}}}$. The analysis shows that the Eq. (31) is valid for $\rho > a$, where a is the turning point. The value of a corresponds to the condition $\mathcal{P}(a) = 0$, which results in

$a \sim |m|/k \ll R$. We assume that the parameter ρ_0 satisfies the condition $a \ll \rho_0 \ll R$, hence $1 \sim ka \ll k\rho_0 \ll kR$.

On the other hand, at small distances $\rho \ll R$, the partial potential $U_m \sim \nu^2/\rho^2$, i.e. it describes the free particle in the form (5) with a shifted index:

$$\psi_m^{\mathcal{E}}(\rho) \propto J_{|\nu|}(k\rho), \qquad \text{when } \rho \ll R. \tag{32}$$

For $kR \gg |\nu|$ there is a wide range of values of ρ, namely

$$|\nu|/k \ll \rho \ll R, \tag{33}$$

where we can use the asymptotic expression for the Bessel function (32) in the limit $k\rho \gg |\nu|$ [35]:

$$\psi_m^{\mathcal{E}}(\rho) \propto \frac{1}{\sqrt{\rho}} \cos\Big(k\rho - \frac{|\nu|\pi}{2} - \frac{\pi}{4} + \frac{4\nu^2 - 1}{8k\rho}\Big). \tag{34}$$

In the range (33) the solutions (31) and (34) agree due to an overlap in the entire range of parameters, so one can derive the phase χ_0 in the WKB–solution (31),

$$\chi_0 = k\rho_0 - \frac{|\nu|\pi}{2} - \frac{\pi}{4} + \frac{4\nu^2 - 1}{8k\rho_0} .$$

Therefore, we are able to calculate the short–wavelength asymptotic expression for the scattering wave phase shift by the asymptotic expansion of the WKB–solution (31):

$$\begin{aligned} \delta_m(k) = \lim_{\rho\to\infty} \Bigg(\int_{\rho_0}^{\rho} \mathcal{P}(\rho')d\rho' + \chi_0 - k\rho \\ + \frac{|m|\pi}{2} + \frac{\pi}{4} - \frac{4m^2 - 1}{8k\rho} \Bigg). \end{aligned} \tag{35}$$

Under the condition $k\rho \gg 1$, the WKB–integral in (35) can be calculated in the leading approximation in $1/k\rho$,

$$\int_{\rho_0}^{\rho} \mathcal{P}(\rho')d\rho' \approx k(\rho - \rho_0) - \frac{1}{2k} \int_{\rho_0}^{\rho} \mathcal{U}_{\text{eff}}(\rho')d\rho'.$$

As result, the scattering phase shift for large wave numbers, $k \gg 1/R$, has the form

$$\delta_m(k) = \pi \cdot \frac{|m| - |\nu|}{2} - \frac{1}{2k} \int_0^{\infty} \Delta U_m(\rho')d\rho', \tag{36}$$

which contains a general, so–called eikonal dependence $\delta \propto 1/k$. The potential ΔU_m,

$$\Delta U_m(\rho) = U_m(\rho) - \frac{\nu^2}{\rho^2} = V(\rho) + \frac{m^2 - \nu^2}{\rho^2}$$

has no singularities at the origin, $\lim_{\rho=0} \rho^2 \Delta U_m(\rho) = 0$. Note that the scattering phase shift does not tend to zero for $k \to \infty$, but at some finite value $(\pi/2) \cdot (|m| - |\nu|)$. This feature is caused by the singularity of the potential at the origin.

Let us discuss the Levinson theorem. Before proceeding to a formal proof (see the next section), it will be useful to present a heuristic argument. We consider the scattering problem near the threshold, $k = 0$. Let us suppose that the potential well is so deep that bound states for $\mathcal{E} \lesssim 0$ can be described by the WKB approximation (31) with the phase shifts given by (35). The WKB–integral in (35) can be calculated in the leading approximation in $k\rho$,

$$\int_{\rho_0}^{\rho} \mathcal{P}(\rho')d\rho' \approx \int_a^b \mathcal{P}(\rho')d\rho' + k\rho + \text{const}, \tag{37}$$

where a and b are the turning points of the quasiclassical motion in the potential $\mathcal{U}_{\text{ef}}$, see Eq. (30). Under such assumptions the Bohr–Sommerfeld quantization rule is valid,

$$\int_a^b \mathcal{P}(\rho')d\rho' = \pi \cdot \left(N_m^{\text{b}} + \gamma\right), \tag{38}$$

where γ depends on the potential's behaviour near the turning points. Note that the bound states are absent, $N_m^{\text{b}} = 0$, for the limiting case of the shallow well, $V \to 0$; this is the case for the Born approximation with the general scattering condition $\delta_m(0) = 0$ [1, 2]. Using Eqs. (37), (38), this results in the phase shift (35) in the form

$$\delta_m(0) = \pi \cdot N_m^{\text{b}}$$

for regular potentials. However, in the case of potentials with inverse square singularity, it should be shifted by $(\pi/2) \cdot (|m| - |\mu|)$, see Eq. (22). Thus, our simple qualitative picture leads to the long–wavelength limit of the phase shift,

$$\delta_m(0) = \pi \cdot \left(N_m^{\text{b}} + \frac{|m| - |\mu|}{2}\right). \tag{39}$$

Using the limiting values for the phase shift, Eqs. (36), (39), one can calculate Levinson's relation:

$$\delta_m(0) - \delta_m(\infty) = \pi \cdot \left(N_m^{\text{b}} + \frac{|\nu| - |\mu|}{2}\right). \tag{40}$$

V. THE LEVINSON THEOREM

Let us enter into a proof of the Levinson theorem. There are three main methods to derive the theorem: the Jost functions method, the Green's functions method, and the

Sturm–Liouville method, which were used for the 3D case, for the details see Ref. [24].

To generalize the Levinson theorem, we use the method of the Green functions, as it was done for regular potentials by Lin [23]. We consider the noncritical case, when the Schrödinger equation has no half bound states.

The idea of Lin's method [23] is to count the number of states in the system by two different ways.

The continuous part of the spectrum is discretized to count the number of scattering states. Therefore, the total (infinite) number of states in the system does not depend on the shape of the potential, it results in

$$\mathrm{Im}\int_{-\infty}^{\infty} d\mathcal{E}\int_0^{\infty}\rho d\rho \left\{G[U_m]-G[U_m^{\text{free}}]\right\}=0, \tag{41a}$$

where $G[U_m]\equiv G_m(\rho,\rho,\mathcal{E};U_m)$ and $G[U_m^{\text{free}}]\equiv G_m(\rho,\rho,\mathcal{E};U_m^{\text{free}})$ are the Green functions with and without potential, respectively; and the retarded Green function is defined by

$$G_m(\rho,\rho',\mathcal{E};U_m)=\sum_{\kappa}\frac{\psi_m^{\mathcal{E}}(\rho)\psi_m^{\mathcal{E}}(\rho')}{\mathcal{E}-\mathcal{E}_{m\kappa}+i\epsilon}\,.$$

In this method, the number of bound states,

$$\pi N_m^{\text{b}}=-\mathrm{Im}\int_{-\infty}^{0} d\mathcal{E}\int_0^{\infty}\rho d\rho \left\{G[U_m]-G[U_m^{\text{free}}]\right\}. \tag{41b}$$

On the other hand, the continuous part of the expression (41a) can be directly calculated without discretization:

$$\mathrm{Im}\int_{0}^{\infty} d\mathcal{E}\int_0^{\infty}\rho d\rho \left\{G[U_m]-G[U_m^{\text{free}}]\right\}=\delta_m(0)-\delta_m(\infty). \tag{41c}$$

Combining Eqs. (41), one can obtain the Levinson theorem in the form (10). However, the method of the Green functions in the form proposed by Lin [23] does not work for singular potentials. The reason is that the difference of Green functions $G[U_m]-G[U_m^{\text{free}}]$ in (41) has a singularity at the origin, hence it is not integrable.

That is why we need to generalize the method for the case of singular potentials. The idea is to compare the required partial potential U_m not with the free particle partial potential U_m^{free}, but with another potential $U_m^{\star}$, which could compensate the singularities of U_m. As we have mentioned before, the number of states does not depend on the shape of the potential. It means that repeating the same proof, Eqs. (41) can be easily generalized for the systems

$G[U_m]$ and $G[U^\star_m]$ with two different potentials U_m and $U^\star_m$:

$$\mathrm{Im}\int_{-\infty}^{\infty} d\mathcal{E}\int_0^\infty \rho d\rho \left\{G[U_m]-G[U^\star_m]\right\}=0, \tag{42a}$$

$$\mathrm{Im}\int_{-\infty}^{0} d\mathcal{E}\int_0^\infty \rho d\rho \left\{G[U_m]-G[U^\star_m]\right\}$$
$$=-\pi\cdot\left(N^{\mathrm{b}}_m-N^{\mathrm{b}\star}_m\right), \tag{42b}$$

$$\mathrm{Im}\int_{0}^{\infty} d\mathcal{E}\int_0^\infty \rho d\rho \left\{G[U_m]-G[U^\star_m]\right\}$$
$$=\delta_m(0)-\delta_m(\infty)-\delta^\star_m(0)+\delta^\star_m(\infty), \tag{42c}$$

where $N^{\mathrm{b}\star}_m$ and $\delta^\star_m(k)$ are the number of bound states and the scattering phase shift for the system with the partial potential $U^\star_m = V^\star + m^2/\rho^2$.

Note that choosing $V^\star = 0$, one can obtain Levinson's relation for the regular potentials in the form of Lin [23], see Eqs. (41), which leads to the Levinson theorem (10).

However, in the case of a singular potential, we need to choose $V^\star$ in the form which has the same singularities as the potential V. To solve the problem we set $U^\star_m = U^{\mathrm{cf}}_m$; hence both partial potentials U_m and the centrifugal potential U^{cf}_m have the same features. Therefore, Eqs. (41) with account of Levinson's relation (28), lead to the following form:

$$\delta_m(0)-\delta_m(\infty)=\pi\cdot\left(N^{\mathrm{b}}_m+\frac{|\nu|-|\mu|}{2}\right), \tag{40$'$}$$

so we reestablish the generalized Levinson theorem in the form (40).

Let us discuss the result. To explain the meaning of the extra term $(\pi/2)\cdot(|\nu|-|\mu|)$ in the generalized Levinson relation (40$'$), let us remind that in the partial wave method the scattering data are classified by the azimuthal quantum number m, which is the strength of the centrifugal potential. In the presence of the potential with an inverse square singularity at the origin like $U_m \sim \nu^2/\rho^2$, the effective singularity strength is shifted by the value $|\nu|-|m|$, which results in a change in the short–wavelength scattering phase shift by $(\pi/2)\cdot(|m|-|\nu|)$. The same situation takes place for the potentials with an inverse square tail at infinity like $U_m \sim \mu^2/\rho^2$. The effective singularity strength is shifted now by the value $|\mu|-|m|$, and the long–wavelength scattering data are changed by $(\pi/2)\cdot(|m|-|\mu|)$. As result, the correction to the Levinson's relation is

$$\pi\cdot\frac{|m|-|\mu|}{2}-\pi\cdot\frac{|m|-|\nu|}{2}=\pi\cdot\frac{|\nu|-|\mu|}{2}.$$

Such a correction looks like a modification in the classification of the scattered states, both at the origin ($\psi_m\to\psi_\nu$), and at the infinity ($\psi_m\to\psi_\mu$). However, we need to stress that the singularity strengths ν and μ can assume any real values, while the quantum number m is always integer.

VI. CONCLUSION

In conclusion, we have established the analogue of the Levinson theorem in the case of two–dimensional scattering for central potentials, which are independent of both the energy and the azimuthal momentum m, but have inverse square singularities and tails.

The presence of m–dependent potentials can essentially change the scattering picture: the symmetry $\delta_m(k) = \delta_{-m}(k)$ is broken, so it is not enough to take into account partial waves with $m \geq 0$ only. As result Levinson's relation (40) has a different form for opposite m. Moreover, the threshold behaviour for the half–bound states changes, so the contribution of the half–bound states in the form (10) may be not adequate.

The generalized Levinson theorem (40′) can be applied to different physical problems. For example, it becomes a central point in the singular inverse method [15], giving a possibility to derive the potential from the scattering phase shift. At the same time it provides a method to count bound states. The method can be used in various 2D field theories with applications to the physics of 2D plasma [30, 31], nuclear physics [32], quantum Hall effect [33], and 2D magnetism [34, 35, 36, 37].

The method of the 2D radial Darboux transformations, considered in the paper, can be applied to the supersymmetric quantum mechanics, e.g. for the problem of phase–equivalent potentials [14, 15, 16, 38, 39], even for energy–dependent potentials [6].

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[1] J. R. Taylor, *Scattering theory: the quantum theory on nonrelativistic collisions* (Wiley, New York, 1972).

[2] R. G. Newton, *Scattering theory of waves and particles* (Springer–Verlag, New York, 1982).

[3] R. G. Newton, J. Math. Phys. **1**, 319 (1960).

[4] L. Rosenberg, Phys. Rev. **A 59**, 1253 (1999).

[5] P. Swan, Nucl. Phys. **46**, 669 (1963).

[6] J. M. Sparenberg and D. Baye, Phys. Rev. **C 61**, 024605 (2000).

[7] T. A. Weber, J. Math. Phys. **40**, 140 (1999).

[8] Z. Q. Ma and G. J. Ni, Phys. Rev. **D 31**, 1482 (1985).

[9] N. Polyatzky, Phys. Rev. Lett. **70**, 2507 (1993).

[10] K. A. Kiers and W. van Dijk, J. Math. Phys. **37 (12)**, 6033 (1996).

[11] F. Vidal and J. Letourneux, Phys. Rev. **C 45**, 418 (1992).

[12] L. Rosenberg, Phys. Rev. **C 58**, 1385 (1998).

[13] L. Rosenberg and L. Spruch, Phys. Rev. **A 54**, 4978 (1996).

[14] J. M. Sparenberg and D. Baye, Phys. Rev. **C 55**, 55 (1997).

[15] J. M. Sparenberg, Phys. Rev. Lett. **85**, 2661 (2000).

[16] B. F. Samsonov and F. Stancu, Phys. Rev. **C 66**, 034001 (2002).

[17] S. H. Dong and Z. Q. Ma, Int. J. Theor. Phys. **39**, 469 (2000).

[18] S. H. Dong, Int. J. Theor. Phys. **39**, 1529 (2000).

[19] S. H. Dong, Eur. Phys. J. **11**, 159 (2000).

[20] Q. G. Lin, Eur. Phys. J. **D 7**, 515 (1999).

[21] S. S. Gousheh, Phys. Rev. **A 65**, 032719 (2002).

[22] E. Farhi, N. Graham, R. L. Jaffe, and H. Weigel, Nuclear Physics **B 595**, 536 (2001).

[23] Q. G. Lin, Phys. Rev. **A 56**, 1938 (1997).

[24] S. H. Dong, X. W. Hou, and Z. Q. Ma, Phys. Rev. **A 58**, 2790 (1998).

[25] S. H. Dong, X. W. Hou, and Z. Q. Ma, Phys. Rev. **A 59**, 995 (1999).

[26] Q. G. Lin, Phys. Rev. **A 5**, 3478 (1998).

[27] G. Barton, J. Phys. **18**, 479 (1985).

[28] D. Bollé, F. Gesztesy, C. Danneels, and S. Wilk, Phys. Rev. Lett. **56**, 900 (1984).

[29] S. H. Dong and Z. Q. Ma, Phys. Rev. **65**, 042717 (2002).

[30] M. E. Portnoi and I. Galbraith, Solid State Commun. **103**, 325 (1997).

[31] M. E. Portnoi and I. Galbraith, Phys. Rev. **B 58**, 3963 (1998).

[32] H. Walliser and G. Holzwarth, Phys. Rev. **B 61**, 2819 (2000).

[33] R. E. Prange and S. M. Girvin, eds., *The quantum Hall effect* (Springer, New York, 1990).

[34] B. A. Ivanov, H. J. Schnitzer, F. G. Mertens, and G. M. Wysin, Phys. Rev. **B 58**, 8464 (1998).

[35] B. A. Ivanov, V. M. Muravyov, and D. D. Sheka, JETP **89**, 583 (1999).

[36] D. D. Sheka, B. A. Ivanov, and F. G. Mertens, Phys. Rev. **B 64**, 024432 (2001).

[37] B. A. Ivanov and G. M. Wysin, Phys. Rev. **B 65**, 134434 (2002).

[38] H. Leeb, S. A. Sofianos, J. M. Sparenberg, and D. Baye, Phys. Rev. **C 62**, 064003 (2000).

[39] J. M. Sparenberg and D. Baye, Phys. Rev. **C 54**, 1309 (1996).