Levinson theorem for Aharonov-Bohm scattering in two dimensions

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We apply the recently generalized Levinson theorem for potentials with inverse-square singularities [Sheka *et al.*, Phys. Rev. A **68**, 012707 (2003)] to Aharonov-Bohm systems in two dimensions (2D). By this theorem, the number of bound states in a given mth partial wave is related to the phase shift and the magnetic flux. The results are applied to 2D soliton-magnon scattering.

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I. INTRODUCTION

In 1949 Levinson [1] established one of the most beautiful results of scattering theory: the Levinson theorem sets up a relation between the number of bound states, N_l^{b}, in a given lth partial wave and the phase shift $\delta_l(k)$: namely, $\delta_l(0)-\delta_l(\infty)=\pi N_l^{\mathrm{b}}$.

Ten years later, in 1959, Aharonov and Bohm [2] discovered the global properties of the magnetic flux. Nowadays the Aharonov-Bohm (AB) effect is often involved to understand different quantum-mechanical phenomena [3].

The aim of this paper is to generalize the Levinson theorem to systems which exhibit AB effects. We denote such systems as AB systems. Recently, the analog of the Levinson theorem was established by Lin [4] for the simplest AB system with constant magnetic flux Φ. Here we establish a more general relation, valid for a magnetic field with a vector potential of the form

$$\boldsymbol{A}(\boldsymbol{r})=\frac{\Phi(\rho)}{2\pi}\boldsymbol{\nabla}\chi\equiv\frac{\Phi(\rho)}{2\pi\rho}\boldsymbol{e}_{\chi},$$

$$\Phi(0)=2\pi\alpha,\quad \Phi(\infty)=2\pi\beta. \tag{1}$$

Here ρ and χ are the polar coordinates in two spatial dimensions.

The paper is organized as follows. In Sec. II we formulate the scattering problem for the AB systems (1). We prove the Levinson theorem for the simplest so-called *centrifugal* AB model in Sec. III A. The general form of the theorem is established in Sec. III B. We compare our results with those for the conventional AB system in Sec. IV and discuss the physical meaning of the extra term in the generalized theorem. In Sec. V we apply our results to two-dimensional (2D) magnetic systems. Namely, we consider the soliton-magnon scattering, which can be described in the framework of AB scattering of the general form (1). Concluding remarks are presented in Sec. VI.

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II. SCATTERING PROBLEM FOR THE AB SYSTEM: NOTATIONS AND PARTIAL-WAVE EXPANSION

Let us consider the Schrödinger-like equation for a spinless particle in a magnetic field in two dimensions:

$$(-i\boldsymbol{\nabla}-\boldsymbol{A})^2\Psi+V(\boldsymbol{r})\Psi=i\partial_t\Psi. \tag{2}$$

We will consider a central (axially symmetric) potential $V(\boldsymbol{r})=V(\rho)$ and a magnetic vector potential in the form (1). Such a form of the magnetic field is typical for the Aharonov-Bohm effect; it corresponds to the magnetic induction

$$\boldsymbol{B}=\boldsymbol{\nabla}\times\boldsymbol{A}=\boldsymbol{e}_z\left[\frac{\Phi'(\rho)}{2\pi\rho}+\Phi(\rho)\delta(\boldsymbol{r})\right].$$

Thus, the magnetic field has a singular point at the origin (vortex line). The total magnetic flux is $\int B_z \mathrm{d}^2x=\Phi(\infty)$.

We will denote the systems with the above-mentioned potentials as AB systems. For such systems it is possible to apply the standard partial-wave expansion, using the ansatz

$$\Psi(\boldsymbol{r},t)=\sum_{m=-\infty}^{\infty}\psi_m^{\mathcal{E}}(\rho)\exp(im\chi-i\mathcal{E}t), \tag{3}$$

where $\{m,\mathcal{E}\}$ is the complete set of eigennumbers and $\mathcal{E}$ and m are the energy and the azimuthal quantum number, respectively. Each partial wave $\psi_m^{\mathcal{E}}$ is an eigenfunction of the spectral problem

$$H\psi_m^{\mathcal{E}}(\rho)=\mathcal{E}\psi_m^{\mathcal{E}}(\rho) \tag{4a}$$

for the 2D radial Schrödinger operator $H=-\nabla_\rho^2+U_m(\rho)$ with the partial potential

$$U_m(\rho)=V(\rho)+\frac{\left[m-\dfrac{\Phi(\rho)}{2\pi}\right]^2}{\rho^2}. \tag{4b}$$

Let us formulate the scattering problem. A continuum spectrum exists for $\mathcal{E}>0$. Note that the eigenfunctions for the free particle, $V(\rho)=\Phi(\rho)=0$, have the form

$$\psi_m^{\text{free}}(\rho) \propto J_m(k\rho), \quad k = \sqrt{\mathcal{E}} > 0, \tag{5}$$

where k is a "radial wave number" and J_m is a Bessel function. Free eigenfunctions like ψ_m^{free} play the role of partial cylinder waves of the plane wave:

$$\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\mathcal{E}t) = \sum_{m=-\infty}^{\infty} i^m J_m(k\rho)e^{im\chi - i\mathcal{E}t}. \tag{6}$$

The behavior of the eigenfunctions in the potentials $V(\rho)$ and $\Phi(\rho)$ can be analyzed at large distances from the origin, $\rho \gg R$, where R is a typical range of the potentials. In view of the asymptotic behavior $U_m(\rho) \sim m^2/\rho^2$, which is valid for fast decreasing potentials $V(\rho)$ and $\Phi(\rho)$, in the leading approximation in $1/\rho$ we have the usual result

$$\psi_m^{\mathcal{E}} \propto J_{|m|}(k\rho) + \sigma_m(k)Y_{|m|}(k\rho), \tag{7}$$

where Y_m is a Neumann function. The quantity $\sigma_m(k)$ stems from the scattering; it can be interpreted as the scattering amplitude. In the limiting case $k\rho \gg |m|$ it is convenient to consider the asymptotic form of Eq. (7),

$$\psi_m^{\mathcal{E}} \propto \frac{1}{\sqrt{\rho}} \cos\left(k\rho - \frac{|m|\pi}{2} - \frac{\pi}{4} + \delta_m(k)\right),$$

where the scattering phase or the phase shift $\delta_m(k) = -\arctan \sigma_m(k)$. The phase shift contains all information about the scattering process. In particular, we give the general solution of the scattering problem for the plane wave (6). With Eqs. (3) and (7), the asymptotic solution of the Schrödinger-like equation (2) for $\rho \gg R$ can be written as

$$\Psi(\boldsymbol{r},t) = \sum_{m=-\infty}^{\infty} C_m[J_{|m|}(k\rho) + \sigma_m(k)Y_{|m|}(k\rho)]\exp(im\chi - i\mathcal{E}t), \tag{8}$$

where C_m are constants. To solve the scattering problem for the plane wave let us choose the constants C_m by comparing Eq. (8) with the expansion (5) for the free motion. Using the asymptotic forms for the cylinder functions in the region $\rho \gg 1/k$, we obtain

$$\Psi(\boldsymbol{r},t) = e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\mathcal{E}t} + \mathcal{F}(\chi)\frac{e^{ik\rho - i\mathcal{E}t}}{\sqrt{\rho}},$$

$$\mathcal{F}(\chi) = \frac{\exp(-i\pi/4)}{\sqrt{2\pi k}} \sum_{m=-\infty}^{\infty} (e^{2i\delta_m} - 1)e^{im\chi}. \tag{9}$$

The total scattering cross section is given by the expression

$$\mathfrak{S}^{\text{tot}} = \int_0^{2\pi} |\mathcal{F}|^2 d\chi = \sum_{m=-\infty}^{\infty} \mathfrak{S}_m, \tag{10}$$

where $\mathfrak{S}_m = (4/k)\sin^2\delta_m$ are the partial scattering cross sections.

III. LEVINSON THEOREM FOR THE AB MODEL

For regular 2D potentials $V(\rho)$ without magnetic field $[\Phi(\rho)=0]$, the 2D analog of the Levinson theorem has the form [5–7]

$$\delta_m(0) - \delta_m(\infty) = \pi(N_m^{\text{b}} + N_m^{\text{hb}}\delta_{|m|,1}). \tag{11}$$

Here N_m^{b} is the number of bound states in a given mth partial wave and N_m^{hb} is the number of half-bound states (recall that a zero-energy state is called a half-bound state if its wave function is finite, but does not decay fast enough at infinity to be square integrable).

Here all partial potentials $U_m(\rho)$ satisfy the asymptotic conditions

$$\lim_{\rho=0} \rho^2 U_m(\rho) = m^2, \tag{12a}$$

$$\lim_{\rho=\infty} \rho^2 U_m(\rho) = m^2, \tag{12b}$$

which provide a regular behavior at the origin and fast decaying at infinity.

The presence of the nonlocal magnetic field can break the asymptotic conditions (12). Namely, if the field does not vanish at the origin, $\Phi(0) = 2\pi\alpha \neq 0$, the asymptotic condition (12a) is broken. In the same way, a not vanishing field at infinity, $\Phi(\infty) = 2\pi\beta \neq 0$, breaks the asymptotic condition (12b). There appear inverse-square singularities in the effective partial potential at the origin or in the inverse-square tail at infinity. The standard Levinson theorem fails for this case [4,8], and some generalization is needed.

Before we discuss the general case, let us consider the simplest AB model, which nevertheless contains the main features of the problem.

A. Simplest "centrifugal" AB model

We start with vector potentials of the form

$$\boldsymbol{A}(\boldsymbol{r}) = \begin{cases} \alpha \boldsymbol{\nabla}\chi & \text{when } \rho < R, \\ \beta \boldsymbol{\nabla}\chi & \text{otherwise,} \end{cases} \tag{13}$$

where α and β are nonzero constants. For this simple model the potential $V(\rho) \equiv 0$, so the effective partial potentials (4b) for the correspondent spectral problem (4) can be rewritten as follows:

$$U_m(\rho) = \begin{cases} \dfrac{\nu^2}{\rho^2} & \text{when } \rho < R, \\ \dfrac{\mu^2}{\rho^2} & \text{otherwise,} \end{cases}$$

$$\nu \equiv m - \alpha, \quad \mu \equiv m - \beta.$$

The scattering problem for this so-called *centrifugal model* has an exact solution (see [8])

$$\delta_m^{\rm cf}(k) = \frac{|m|-|\mu|}{2}\pi - \arctan \tilde{\sigma}_\mu^{\rm cf}(\varkappa \equiv kR),$$

$$\tilde{\sigma}_\mu^{\rm cf}(\varkappa) = \frac{J'_{|\nu|}(\varkappa)J_{|\mu|}(\varkappa) - J'_{|\mu|}(\varkappa)J_{|\nu|}(\varkappa)}{J_{|\nu|}(\varkappa)Y'_{|\mu|}(\varkappa) - J'_{|\nu|}(\varkappa)Y_{|\mu|}(\varkappa)}.$$

Using the asymptotic form of the cylinder functions, one can easily derive the Levinson relation for the centrifugal model [8]:

$$\delta_m^{\rm cf}(0) - \delta_m^{\rm cf}(\infty) = \frac{\pi}{2}(|\nu| - |\mu|). \tag{14}$$

As an example we consider a solenoid of zero radius with constant magnetic flux Φ_0 and returned flux uniformly distributed on the surface of a cylinder at radius R; the vector potential of such a system is [9,10]

$$\boldsymbol{A} = \begin{cases} \dfrac{\Phi_0}{2\pi\rho}\boldsymbol{e}_\chi & \text{when } \rho < R, \\ 0 & \text{otherwise}. \end{cases} \tag{15}$$

The magnetic induction $\boldsymbol{B}=\frac{\Phi_0}{2\pi\rho}[\delta(\rho)-\delta(\rho-R)]\boldsymbol{e}_z$ consists of the usual AB flux line at $\rho=0$ and an infinitely thin magnetic field shell at $\rho=R$. Identifying the parameters $\alpha=\Phi_0/2\pi$ and $\beta=0$, one can rewrite the Levinson relation (14) as follows:

$$\delta_m(0) - \delta_m(\infty) = \frac{\pi}{2}(|m - \Phi_0/2\pi| - |m|). \tag{16}$$

Note that the Levinson relation takes nonzero values for any nonvanishing AB field flux Φ_0, which can take also an integer value. In particular, if $m > \Phi_0/2\pi > 0$, the Levinson relation is equal to $-\Phi_0/4$.

B. Levinson theorem for general AB systems

Let us discuss the case of the general AB system with the vector potential of the form (1). We suppose that the particle potential $V(\rho)$ is less singular than ρ^{-2} at the origin and decays faster than ρ^{-2} at infinity. Then the partial potential (4b) satisfies the asymptotic conditions

$$U_m(\rho) \sim \begin{cases} \dfrac{\nu^2}{\rho^2}, & \text{when } \rho \to 0, \\ \dfrac{\mu^2}{\rho^2}, & \text{when } \rho \to \infty, \end{cases} \tag{17}$$

where $\nu=m-\alpha$ and $\mu=m-\beta$. In the presence of magnetic flux at least one of the parameters α and β has a nonzero value. This breaks the regular asymptotic conditions (12). In the general case there appears an effective potential, which has an inverse-square singularity at the origin ($\nu \neq m$) and an inverse-square tail at infinity ($\mu \neq m$). The Levinson theorem for such singular potentials was generalized in our recent paper [8]. Namely, when an effective partial potential has the asymptotic behavior (17), the generalized Levinson theorem [8] reads

$$\delta_m(0) - \delta_m(\infty) = \pi\left(N_m^{\rm b} + \frac{|\nu| - |\mu|}{2}\right). \tag{18}$$

Identifying the parameters ν and μ one can rewrite the Levinson relation in the following form:

$$\delta_m(0) - \delta_m(\infty) = \pi\left(N_m^{\rm b} + \frac{|m-\alpha| - |m-\beta|}{2}\right). \tag{19}$$

IV. DISCUSSION

The Levinson theorem (19) establishes a relation between the number of bound states in a given mth partial wave, total phase shift, and magnetic flux.

Let us discuss the physical meaning of the extra term

$$\frac{\pi}{2}(|m-\alpha| - |m-\beta|) \tag{20}$$

in the Levinson relation. This term results from the long-range behavior of the AB potential. The singular behavior of the AB potential at the origin creates a "vorticity" α, which induces wave functions with m greater (smaller) than α to go around the origin in the counterclockwise (clockwise) direction. Thus the short-wavelength scattering data are shifted by $(\pi/2)(|m|-|m-\alpha|)$. The same situation takes place for AB potentials with a long-range tail, which creates a "vorticity" β, and the long-wavelength scattering data are changed by $(\pi/2)(|m-\beta|-|m|)$. As a result, the correction to the Levinson relation takes the form (20).

Let us compare our results with those for the conventional AB system [2,11]:

$$\boldsymbol{A} = \begin{cases} \dfrac{1}{2}B\rho\boldsymbol{e}_\chi & \text{when } \rho < R, \\ \dfrac{BR^2}{2\rho}\boldsymbol{e}_\chi & \text{otherwise}. \end{cases} \tag{21}$$

Such a field produces a constant magnetic induction $\boldsymbol{B}=B\boldsymbol{e}_z$ inside a cylinder of radius R and provides an empty induction outside. The Levinson relation for this case reads

$$\delta_m(0) - \delta_m(\infty) = \frac{\pi}{2}(|m| - |m-\beta|), \quad \beta = \frac{1}{2}BR^2,$$

in agreement with exact results [12,13].

Let us recall that the AB total scattering cross section vanishes when $\beta \in \mathbb{Z}$. Nevertheless, any AB field changes the standard Levinson relation, even when $\beta \in \mathbb{Z}$. Due to the nonlocality of AB potentials, the total phase shifts do not go to zero with increasing $|m|$. To treat such a singularity regularization is usually involved [12] to determine the total scattering amplitude (9) or the total scattering cross section (10). The same picture takes place not only for the conventional AB system (21), but also for the general case (1). An exception is a simple AB system with

$$\boldsymbol{A}(\boldsymbol{r}) = \alpha \boldsymbol{\nabla} \chi. \tag{22}$$

In this case one has a standard Levinson relation in the form $\delta_m(0)-\delta_m(\infty)=\pi N_m^{\mathrm{b}}$. One should note that nevertheless each scattering state $\delta_m(k)$ corresponds to a given general angular momentum $\nu=|m-\alpha|$. The Levinson theorem for this particular case was first obtained by Lin [4].

The Levinson relations should be modified for the critical case when half-bound states occur. The Levinson theorem for the system with possible half-bound states was considered first by Bollé *et al.*[5] and reestablished later by another method by Dong *et al.* [7]. Without magnetic field the Levinson relation has the form of Eq. (11), so the half-bound states affect in the same way the two modes with $m=\pm 1$. The presence of the magnetic field breaks the symmetry $\delta_m(k)=\delta_{-m}(k)$, and in the general case the contribution of the half-bound states in the form (11) cannot be adequate. However, for the particular case (22), the problem can be solved [4].

If the particle potential $V(\rho)$ has an inverse-square singularity or an inverse-square tail, then the Levinson theorem in the form (19) fails. Instead, one has to calculate the effective intensities ν and μ of the singularities in the partial potential as follows:

$$\nu^2=\lim_{\rho=0}\rho^2 U_m(\rho), \qquad \mu^2=\lim_{\rho=\infty}\rho^2 U_m(\rho), \tag{23}$$

and then one obtains the Levinson theorem in the form (18).

V. APPLICATIONS TO MAGNETISM: SCATTERING ON A MAGNETIC SOLITON IN 2D ISOTROPIC MAGNETS

All mentioned above results can be applied to a wide class of AB systems. Here we do not consider a quantum-mechanical example of the general AB-scattering system. Namely, we apply our results to the description of the soliton-magnon interaction in a 2D magnet. Note that it is possible to apply the quantum AB theorem to a classical system, because magnons in a magnet can be formally described by a Scrödinger-like equation with an effective magnetic field in the form which is typical for AB systems.

We consider the model of a 2D isotropic Heisenberg ferromagnet, where the elementary linear excitations of the spin system (magnons) can coexist together with nonlinear ones (solitons). In terms of the angular variables for the normalized magnetization $\boldsymbol{m}=(\sin\theta\cos\phi;\sin\theta\sin\phi;\cos\theta)$, the structure of the simplest nonlinear excitation, the so-called Belavin-Polyakov soliton, is described by the formulas [14]

$$\tan\frac{\theta_0(\rho)}{2}=\left(\frac{R}{\rho}\right)^{|q|}, \qquad \phi_0=\varphi_0+q\chi.$$

Here $q\in\mathbb{Z}$ is the topological charge of the soliton and R and φ_0 are arbitrary parameters.

To analyze the soliton-magnon interaction, one considers small oscillations of the magnetization (θ,ϕ) on the background of the soliton (θ_0,ϕ_0). These oscillations can be described in terms of the complex-valued "wave function" $\psi=\theta-\theta_0+i\sin\theta_0(\phi-\phi_0)$. The linearized equations have the form of the Schrödinger-like equation (2) with an effective potential [15,16]

$$V(\rho)=-\frac{q^2}{\rho^2}\sin^2\theta_0$$

and an effective magnetic field in the form

$$\boldsymbol{A}(\boldsymbol{r})=\frac{\Phi(\rho)}{2\pi}\boldsymbol{\nabla}\chi, \quad \Phi(\rho)=-2\pi q\cos\theta_0(\rho),$$

$$\Phi(0)=2\pi q, \quad \Phi(\infty)=-2\pi q.$$

The partial potential (4b) has the form [15]

$$U_m(\rho)=\frac{m^2+2mq\cos\theta_0(\rho)+q^2\cos 2\theta_0(\rho)}{\rho^2}.$$

Using Eq. (23), one can calculate the intensities of the inverse-square singularities:

$$\nu=|m-q|, \quad \mu=|m+q|.$$

The Levinson theorem reads

$$\delta_m(0)-\delta_m(\infty)=\pi\left(N_m^{\mathrm{b}}+N_m^{\mathrm{hb}}+\frac{|m-q|-|m+q|}{2}\right). \tag{24}$$

As found by Ivanov [17], the soliton with a topological charge q has $2|q|$ internal zero-frequency modes, when $m\in[-q+1;q]$. Namely, modes with $m\in[-q+2;q]$ form bound state, while the mode with $m=-q+1$ is the half-bound state. Finally the Levinson theorem for the soliton-magnon scattering takes the form (we chose $q>0$)

$$\frac{\delta_m(0)-\delta_m(\infty)}{\pi}=\begin{cases} q & \text{when } m\leqslant -q,\\ 1-m & \text{when } -q<m\leqslant q,\\ -q & \text{when } m>q.\end{cases}$$

This result agrees with our previous analytical and numerical calculation for the soliton with $q=1$; see Ref. [15].

Note that the phase shift varies in a wide range, so it *cannot be described*, not even approximately, in the framework of the Born approximation. It was the source of numerous inconsistencies between previous attempts to calculate the soliton-magnon interaction in magnets [18–20]. The reason is that due to the nonlocality of the AB magnetic field, the perturbative Born approximation is not adequate for the AB scattering [12,21].

VI. CONCLUSION

In conclusion, we have applied our recent results [8] for the scattering in a singular potential to AB systems and established a generalization of the Levinson theorem. The theorem constructs the relation between the number of bound states N_m^{b} in a given mth partial wave, the total phase shift $\delta_m(0)-\delta_m(\infty)$ of the scattering state, and the magnetic flux Φ. When the magnetic flux parameter takes different values at the origin and at infinity, $\alpha=\Phi(0)/2\pi$ and $\beta=\Phi(\infty)/2\pi$, the Levinson relation takes the form of Eq. (19). The total phase shift can be treated as a counter for the bound states.

The generalized Levinson theorem (19) can be applied to different AB systems, including quantum Hall systems, superconductors, and so forth [3]. The method can be used not only for quantum-mechanical AB systems. In particular, we have verified the theorem for the case of the soliton-magnon interaction in the 2D isotropic Heisenberg model.

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