Supplemental Material for “Control of magnetic response in curved stripes by tailoring cross section”

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Supplemental Material provides details on analytical calculations of main aspects of the magnetization dynamics in stripes with varying cross section.

SM-1. MODEL

We model ferromagnet wire/stripe, which central curve $\boldsymbol{\gamma} = \{x, y, z\}$ can be parameterized by the arc length s. Following the notations of the main text we introduce dimensionless coordinate along this curve $\xi = s/\ell$ with $\ell = \sqrt{A/(4\pi M_S^2)}$ being the exchange length. The local reference frame at $\boldsymbol{\gamma}(\xi)$ can be introduced using a Frenet trihedron $\{\boldsymbol{e}_\mathrm{T}, \boldsymbol{e}_\mathrm{N}, \boldsymbol{e}_\mathrm{B}\}$, which defines tangential, normal and binormal directions [1]. Differential properties of $\boldsymbol{\gamma}$ are described by Frenet–Serret formula

$$\boldsymbol{e}_i' = \boldsymbol{\varpi} \times \boldsymbol{e}_i, \qquad i = \mathrm{T,N,B}, \tag{S1}$$

where the Darboux vector $\boldsymbol{\varpi} = \sigma \boldsymbol{e}_\mathrm{T} + \varkappa \boldsymbol{e}_\mathrm{B}$ with $\varkappa = \kappa\ell$ and $\sigma = \tau\ell$ being the reduced curvature and torsion of the curve, respectively [2] (κ and τ are dimensional curvature and torsion).

Let us defines a sample as a space domain

$$\boldsymbol{r}(\xi, \zeta_1, \zeta_2) = \boldsymbol{\gamma}(\xi) + \zeta_1 \boldsymbol{e}_\mathrm{N}(s) + \zeta_2 \boldsymbol{e}_\mathrm{B}(s). \tag{S2}$$

We consider both wires and stripes. In case of curved stripe, the perpendicular cross section is parameterized by $\zeta_1 \in [-w(\xi)/2, w(\xi)/2]$ and $\zeta_2 \in [-h(\xi)/2, h(\xi)/2]$ with $h(\xi)$ and $w(\xi)$ being the coordinate-dependent stripe height and width; the cross-section area $S(\xi) = w(\xi)h(\xi)$. Note that the deformation does not break an assumption of one-dimensionality of magnetization. Besides, to avoid an overlap between the neighboring windings of the stripe and reduce the influence of the demagnetization field, the distance between them must be bigger than the stripe width w.

We describe magnetic properties of the stripe using the model of biaxial magnet with the energy density $W = W^\mathrm{X} + W^\mathrm{A}$, see the main text. Here we discuss the biaxial anisotropy, $W^\mathrm{A} = -K_\mathrm{T}^\mathrm{eff}(\boldsymbol{m}\cdot\boldsymbol{e}_\mathrm{T})^2 + K_\mathrm{B}^\mathrm{eff}(\boldsymbol{m}\cdot\boldsymbol{e}_\mathrm{B})^2$. Constants of effective anisotropies, $K_\mathrm{T}^\mathrm{eff} = K_\mathrm{T} + 4\pi M_s^2 k_\mathrm{T}^\mathrm{ms}$ and $K_\mathrm{B}^\mathrm{eff} = K_\mathrm{B} + 4\pi M_s^2 k_\mathrm{B}^\mathrm{ms}$ incorporate magnetocrystalline anisotropy coefficients K_T and K_B as well as magnetostatic contributions, k_T^ms and k_B^ms. In case of straight and uniformly magnetized stripe with rectangular cross section, the magnetostatic energy is well-known [3–5] to be reduced to effective shape anisotropy [5] with coefficients

$$\begin{aligned} k_\mathrm{T}^\mathrm{ms} &= \frac{\frac{1-a^2}{2a}\ln\left(1+a^2\right) + a\ln a + 2\arctan\frac{1}{a}}{2\pi}, \\ k_\mathrm{B}^\mathrm{ms} &= \frac{1}{2} - 2k_\mathrm{T}^\mathrm{ms}, \quad a = w/h \geq 1. \end{aligned} \tag{S3}$$

For thin, narrow, and curved stripes the approximation of the shape anisotropy is used also for inhomogeneous magnetization states [6], including domain walls [7, 8]. In the limit case of square ($w/h = 1$) or circular cross sections, the magnetostatic-shape-induced anisotropy coefficients (S3) are simplified to $k_\mathrm{T}^\mathrm{ms} = 1/4$ and $k_\mathrm{B}^\mathrm{ms} = 0$, which is a well known result [3, 9] including the case of curvilinear wires [10]. In case of the curved stripe with varying cross section, its aspect ratio becomes coordinate dependent, $a = a(\xi)$, hence anisotropy coefficients becomes, in general, also coordinate dependent, $K_\mathrm{T}^\mathrm{eff} = K_\mathrm{T}^\mathrm{eff}(\xi)$ and $K_\mathrm{B}^\mathrm{eff} = K_\mathrm{B}^\mathrm{eff}(\xi)$.

In the currents study we limit ourselves by planar stripes (i.e. stripes with zero torsion). Torsion induced effects in stripes with coordinate-dependent cross section and curvature will be considered somewhere else.

SM-2. MAGNETIZATION TILTING BY THE GRADIENT OF CROSS SECTION

Here we discuss the case of strong anisotropy. We suppose that the magnetic texture will not deviate significantly from the equilibrium state given by the anisotropy. The geometry–induced anisotropy and DMI act as a source of the emergent geometry–induced magnetic field [11, 12], which influences even the equilibrium state magnetization: an assumed ground state becomes a subject of further modifications due to curvilinear effects [11–14]. Assuming small deviations from the tangential directions, we utilize the angular parametrization

$$\boldsymbol{m}_\mathrm{BT} = \boldsymbol{e}_\mathrm{T}\sin\Theta\cos\Phi + \boldsymbol{e}_\mathrm{N}\sin\Theta\sin\Phi + \boldsymbol{e}_\mathrm{B}\cos\Theta, \tag{S4}$$

where Θ characterizes the deviation from the binormal direction and Φ describes the deviation from the tangential direction in the osculating TN plane. Then the

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energy density (1) reads

$$\begin{aligned}\mathscr{E} = \mathcal{S}(\xi)\Big\{ &\left(\Theta' - \sigma\sin\Phi\right)^2 \\ &+\left[\sin\Theta(\Phi'+\varkappa) - \sigma\cos\Theta\cos\Phi\right]^2 \\ &-k_{\mathrm{T}}\sin^2\Theta\cos^2\Phi + k_{\mathrm{B}}\cos^2\Theta\Big\}.\end{aligned} \tag{S5}$$

The strictly tangential magnetisation corresponds to $\Theta_{\mathrm{T}} = \pi/2$ and $\cos\Phi_{\mathrm{T}} = \mathfrak{C} = \pm 1$. Here $\mathfrak{C} = +1$ corresponds to parallel orientation along $\boldsymbol{e}_{\mathrm{T}}$ and $\mathfrak{C} = -1$ corresponds to the antiparallel. Assuming small deviations from this direction and putting $\Theta = \Theta_{\mathrm{T}} + \vartheta$, $\phi = \Phi_{\mathrm{T}} + \varphi$ with $|\vartheta|, |\varphi| \ll 1$, we can present the energy density (S5) as presented in Eq. (2).

SM-3. DOMAIN WALL FOR PLANAR GEOMETRIES

Magnetization dynamics of this system is studied by means of phenomenological Landau–Lifshitz–Gilbert equations

$$-\sin\theta\dot{\theta} = \frac{\delta\mathscr{E}}{\delta\phi} + \alpha\sin^2\theta\dot{\phi}, \quad \sin\theta\dot{\phi} = \frac{\delta\mathscr{E}}{\delta\theta} + \alpha\dot{\theta}, \tag{S6}$$

where constant α is a Gilbert damping coefficient, overdot indicates the the derivative with respect to dimensionless time $\bar{t} = \omega_0 t$, where $\omega_0 = 4\pi\gamma_0 M_{\mathrm{S}}$ as $\Omega = \omega/\omega_0$. These equations of motion can be derived from the Lagrangian

$$\mathscr{L} = \mathscr{G} - \mathscr{E}, \quad \mathscr{G} = -\int\limits_{-\infty}^{+\infty} \phi\sin\theta\dot{\theta}\mathcal{S}\mathrm{d}\xi \tag{S7}$$

and Rayleigh dissipative function

$$\mathscr{R} = \frac{\alpha}{2}\int\limits_{-\infty}^{+\infty}\left[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right]\mathcal{S}\mathrm{d}\xi \tag{S8}$$

as Lagrange–Rayleigh equations

$$\frac{\delta\mathscr{L}}{\delta\psi_\nu} - \partial_{\bar{t}}\frac{\delta\mathscr{L}}{\delta\dot{\psi}_\nu} = \frac{\delta\mathscr{R}}{\delta\dot{\psi}_\nu}, \qquad \psi_\nu \in \{\theta, \phi\}. \tag{S9}$$

Let us limit our consideration by planar geometries, when the torsion is absent, $\sigma = 0$. The central line of this stripe can be described by a planar curve $\boldsymbol{\gamma}(\xi) = \gamma_x(\xi)\hat{\boldsymbol{x}} + \gamma_y(\xi)\hat{\boldsymbol{y}}$. It is convenient to measure the polar angle from the anisotropy axis using

$$\boldsymbol{m}_{\mathrm{TN}} = \boldsymbol{e}_{\mathrm{T}}\cos\theta + \boldsymbol{e}_{\mathrm{N}}\sin\theta\cos\phi + \boldsymbol{e}_{\mathrm{B}}\sin\theta\sin\phi. \tag{S10}$$

Then the normalized energy of the magnetization texture in the planar geometry reads

$$\begin{aligned}\mathscr{E}[\theta,\phi] = \int\limits_{-\infty}^{\infty}\mathrm{d}\xi\mathcal{S}\Big[&\left(\theta' + \varkappa\cos\phi\right)^2 + \big(\phi'\sin\theta \\ &- \varkappa\cos\theta\sin\phi\big)^2 + \sin^2\theta\left(k_{\mathrm{T}} + k_{\mathrm{B}}\sin^2\phi\right)\Big].\end{aligned} \tag{S11}$$

Equilibrium magnetization textures can be found by minimization of this energy, which results in following set of equations

$$\begin{aligned}&\theta'' - \sin\theta\cos\theta\left[\phi'^2 + \left(k_{\mathrm{B}} - \varkappa^2\right)\sin^2\phi + k_{\mathrm{T}}\right] \\ &\; - 2\phi'\varkappa\sin^2\theta\sin\phi + \varkappa\cos\phi\left(\frac{\mathcal{S}'}{\mathcal{S}} + \frac{\varkappa'}{\varkappa}\right) + \theta'\frac{\mathcal{S}'}{\mathcal{S}} = 0,\end{aligned} \tag{S12a}$$

$$\begin{aligned}&\phi'' + 2\theta'\left[\phi'\cot\theta + \varkappa\sin\phi\right] + \phi'\frac{\mathcal{S}'}{\mathcal{S}} \\ &\; - \left(k_{\mathrm{B}} - \varkappa^2\right)\sin\phi\cos\phi - \varkappa\cot\theta\sin\phi\left(\frac{\mathcal{S}'}{\mathcal{S}} + \frac{\varkappa'}{\varkappa}\right) = 0.\end{aligned} \tag{S12b}$$

Eq. (S12b) has solutions $\cos\phi_0 = \mathscr{C} = \pm 1$. Substitution of this solution into first equation in (S12) results in Eq. (4), which determines the function $\theta(\xi)$.

Now we proceed to collective variable approach, using the Ansatz (5)

$$\cos\theta^{\mathrm{DW}} = -p\tanh\left[\frac{\xi - q(\bar{t})}{\Delta}\right], \quad \phi^{\mathrm{DW}} = \Phi(\bar{t}). \tag{S5$'$}$$

Here $q(\bar{t})$ and $\Phi(\bar{t})$ are collective variables; the DW width Δ is assumed to be a slaved variable [3] i.e., $\Delta = \Delta[q, \Phi]$. First, we derive an effective energy of the domain wall. By substituting an Ansatz (5) into the energy functional (S11), we get the effective energy of the domain wall $\mathscr{E}^{\mathrm{DW}} = \mathscr{E}[\theta^{\mathrm{DW}}, \phi^{\mathrm{DW}}]$, which results in

$$\begin{aligned}\mathscr{E}^{\mathrm{DW}} = 2\mathcal{S}(q)\Big\{&\frac{\hbar_1}{\Delta} + \hbar_4 k_{\mathrm{T}}(q)\Delta + \Delta\left[\hbar_5 k_{\mathrm{B}}(q)\right. \\ &\left.-\hbar_2\varkappa^2(q)\right]\sin^2\Phi + p\pi\hbar_3\varkappa(q)\cos\Phi\Big\}.\end{aligned} \tag{S13}$$

Here and below we use parameters $\hbar_i \equiv \hbar_i(q)$ defined as

$$\begin{aligned}\hbar_1 &= \frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{\mathcal{S}(q+x\Delta)}{\mathcal{S}(q)\cosh^2 x}\mathrm{d}x, \\ \hbar_2 &= \frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{\varkappa^2(q+x\Delta)\mathcal{S}(q+x\Delta)}{\varkappa^2(q)\mathcal{S}(q)\cosh^2 x}\mathrm{d}x, \\ \hbar_3 &= \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{\varkappa(q+x\Delta)\mathcal{S}(q+x\Delta)}{\varkappa(q)\mathcal{S}(q)\cosh x}\mathrm{d}x, \\ \hbar_4 &= \frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{k_{\mathrm{T}}(q+x\Delta)\mathcal{S}(q+x\Delta)}{k_{\mathrm{T}}(q)\mathcal{S}(q)\cosh^2 x}\mathrm{d}x, \\ \hbar_5 &= \frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{k_{\mathrm{B}}(q+x\Delta)\mathcal{S}(q+x\Delta)}{k_{\mathrm{B}}(q)\mathcal{S}(q)\cosh^2 x}\mathrm{d}x.\end{aligned} \tag{S14}$$

Equilibrium values of the domain wall position q_0, its phase Φ_0, and the domain wall width Δ_0 can be found by the energy minimization. The domain wall width $\Delta_0 = \sqrt{\hbar_{1,0}/\left(\hbar_{4,0}k_{\mathrm{T},0}\right)}$, the phase $\cos\Phi_0 = -p\,\mathrm{sgn}\,\varkappa_0$.

The corresponding value of the equilibrium domain wall position q_0 is determined by the equation

$$\frac{\varkappa_0'}{\varkappa_0}+\frac{\mathcal{S}_0'}{\mathcal{S}_0}+\frac{\hbar_{3,0}'}{\hbar_{3,0}}=\frac{\hbar_{1,0}}{\pi\hbar_{3,0}\Delta_0|\varkappa_0|}\left(\frac{\hbar_{1,0}'}{\hbar_{1,0}}+\frac{\hbar_{4,0}'}{\hbar_{4,0}}+\frac{k_{\text{T},0}'}{k_{\text{T},0}}+2\frac{\mathcal{S}_0'}{\mathcal{S}_0}\right), \tag{S15}$$

where $f_0\equiv f(q_0)$, $f_0'\equiv\partial_q f|_{q=q_0}$, $f_{i,0}\equiv f_i(q_0)$, and $f_{i,0}'\equiv\partial_q f_i|_{q=q_0}$. For the sake of simplicity in the following we will consider stripes with constant aspect ratio $a=\text{const}$ which results in the constant anisotropy coefficients k_T and k_B and, besides, equal parameters $\hbar_1=\hbar_4=\hbar_5$.

In the same manner we construct the effective Lagrangian of the domain wall $\mathscr{L}^{\text{DW}}=\mathscr{L}\left[\theta^{\text{DW}},\phi^{\text{DW}}\right]$ and effective dissipative function $\mathscr{R}^{\text{DW}}=\mathscr{R}\left[\theta^{\text{DW}},\phi^{\text{DW}}\right]$:

$$\begin{aligned}\mathscr{L}^{\text{DW}}&=2p\hbar_1\mathcal{S}(q)\Phi\dot{q}-\mathscr{E}^{\text{DW}},\\ \mathscr{R}^{\text{DW}}&=\frac{\alpha\hbar_1}{\Delta}\mathcal{S}(q)\left(\dot{q}^2+\Delta^2\dot{\Phi}^2\right).\end{aligned}\tag{S16}$$

Then effective equations of motion of the domain wall, i.e. the temporal evolution of collective coordinates $q(\bar{t})$ and $\Phi(\bar{t})$ can be described by effective Lagrange–Rayleigh equations

$$\frac{\partial\mathscr{L}^{\text{DW}}}{\partial X_\nu}-\frac{\text{d}}{\text{d}\bar{t}}\frac{\partial\mathscr{L}^{\text{DW}}}{\partial\dot{X}_\nu}=\frac{\partial\mathscr{R}^{\text{DW}}}{\partial\dot{X}_\nu},\qquad X_\nu\in\{q,\,\Phi\}. \tag{S17}$$

Let us consider the case of narrow domain wall, $\Delta\ll\lambda_\varkappa,\lambda_\mathcal{S}$ with $\lambda_\varkappa=\varkappa/\varkappa'$ and $\lambda_\mathcal{S}=\mathcal{S}/\mathcal{S}'$ are inhomogeneity parameters for curvature and cross section deformations, respectively. In this limit case all parameters $\hbar_i=1$, their derivatives $\hbar_i'=0$, hence the energy of the domain wall is reduced to

$$\mathscr{E}^{\text{DW}}\approx2\mathcal{S}(q)\left\{\frac{1}{\Delta}+k_\text{T}\Delta+p\pi\varkappa(q)\cos\Phi+\left[k_\text{B}-\varkappa^2(q)\right]\Delta\sin^2\Phi\right\}. \tag{S18}$$

In this case the general expression for the domain wall position (S15) is reduced to (6a). The effective Lagrangian and the dissipative function of the narrow domain wall

$$\begin{aligned}\mathscr{L}^{\text{DW}}&\approx2\mathcal{S}(q)\left\{p\Phi\dot{q}-\frac{1}{\Delta}-k_\text{T}\Delta-p\pi\varkappa(q)\cos\Phi-\left[k_\text{B}-\varkappa^2(q)\right]\Delta\sin^2\Phi\right\},\\ \mathscr{R}^{\text{DW}}&\approx\frac{\alpha}{\Delta}\mathcal{S}(q)\left(\dot{q}^2+\Delta^2\dot{\Phi}^2\right).\end{aligned}\tag{S19}$$

By substituting the effective Lagrangian and the dissipative function into the Lagrange–Rayleigh equations (S17), one gets effective equations of motion of the narrow wall:

$$\begin{aligned}\alpha\frac{\dot{q}}{\Delta_0}+p\dot{\Phi}&=\left[2\varkappa(q)\Delta_0\sin^2\Phi-p\pi\cos\Phi\right]\varkappa'(q)-\left[\frac{2}{\Delta_0}+p\pi\varkappa(q)\cos\Phi\right]\frac{\mathcal{S}'(q)}{\mathcal{S}(q)},\\ p\frac{\dot{q}}{\Delta_0}-\alpha\dot{\Phi}&=\left[k_\text{B}-\varkappa^2(q)\right]\sin2\Phi-p\pi\frac{\varkappa(q)}{\Delta_0}\sin\Phi.\end{aligned}\tag{S20}$$

Note that the right-hand-side in the first equation determines the internal-to-system driving force. The first term, proportional to the curvature gradient $\varkappa'(q)$ was studied previously. Namely this term is responsible for the domain wall autooscillations [15], its automotion [8]. The newcomer geometrical source is the cross-section gradient, $\mathcal{S}'(q)$. Even in straight stripes with $\varkappa=0$, the gradient of cross section still results in a driving force for the domain wall.

Here, we study linear dynamics of the domain wall in vicinity of the equilibrium position. With this purpose we introduce small deviations in the way $q(\bar{t})=q_0+\mathscr{q}(\bar{t})$ and $\Phi(\bar{t})=\Phi_0+\varphi(\bar{t})$. For the limit case of small curvature and cross section deformation the equations of motion (S20) linearized with respect to the deviations read

$$\begin{aligned}&\begin{bmatrix}\dot{\mathscr{q}}\\ \dot{\varphi}\end{bmatrix}\approx-\begin{bmatrix}\alpha\Omega_\text{g} & -p\Delta_0\Omega_\text{A}\\ p\Omega_\text{g}/\Delta_0 & \alpha\Omega_\text{A}\end{bmatrix}\cdot\begin{bmatrix}\mathscr{q}\\ \varphi\end{bmatrix},\\ &\Omega_\text{g}=\Omega_\varkappa+\Omega_\mathcal{S}+\Omega_{\varkappa\mathcal{S}},\\ &\Omega_\text{A}=2\left(k_\text{B}-\varkappa_0^2\right)+\frac{\pi}{\Delta_0}\left|\varkappa_0\right|,\quad \Omega_\mathcal{S}=2\frac{\mathcal{S}_0''}{\mathcal{S}_0},\\ &\Omega_\varkappa=-\pi\Delta_0\varkappa_0''\,\text{sgn}\,\varkappa_0,\\ &\Omega_{\varkappa\mathcal{S}}=-\pi\Delta_0\frac{\text{sgn}\,\varkappa_0}{\mathcal{S}_0}\left(2\varkappa_0'\mathcal{S}_0'+\varkappa_0\mathcal{S}_0''\right).\end{aligned}\tag{S21}$$

Set of Eqs. (S21) results in a decaying oscillations $\mathscr{q}(\bar{t})=\mathscr{q}_0\sin\Omega\bar{t}\,e^{-\eta\bar{t}}$ and $\varphi(\bar{t})=\varphi_0\cos\Omega\bar{t}\,e^{-\eta\bar{t}}$ with frequency Ω and friction η defined as:

$$\Omega=\sqrt{\Omega_\text{A}\Omega_g},\quad \eta=\frac{\alpha}{2}\left(\Omega_\text{A}+\Omega_g\right). \tag{S22}$$

In more general case $\lambda_\varkappa\Delta_0\in(0,1)$ and $\lambda_\mathcal{S}\Delta_0\in(0,1)$, effective equations of the domain wall oscillations can be derived using Lagrangian and dissipative function in the form (S16) leading to the following expression for the eigenfrequency

$$\begin{aligned}&\Omega=\sqrt{\Omega_\text{A}\Omega_\text{g}},\quad \eta=\frac{\alpha}{2}\left(\Omega_\text{A}+\Omega_g\right),\\ &\Omega_\text{g}=\Omega_\varkappa+\Omega_\mathcal{S}+\Omega_{\varkappa\mathcal{S}},\\ &\Omega_\text{A}=2\left(k_\text{B}-\frac{\hbar_{2,0}}{\hbar_{1,0}}\varkappa_0^2\right)+\frac{\pi}{\Delta_0}\frac{\hbar_{3,0}}{\hbar_{1,0}}\left|\varkappa_0\right|,\\ &\Omega_\varkappa=-\pi\Delta_0\frac{\left(\hbar_3\varkappa\right)_0''}{\hbar_{1,0}}\,\text{sgn}\,\varkappa_0,\quad \Omega_\mathcal{S}=2\frac{\left(\mathcal{S}\hbar_1\right)_0''}{\mathcal{S}_0\hbar_{1,0}},\\ &\Omega_{\varkappa\mathcal{S}}=-\pi\Delta_0\frac{\text{sgn}\,\varkappa_0}{\mathcal{S}_0\hbar_{1,0}}\left[2\left(\hbar_3\varkappa\right)_0'\mathcal{S}_0'+\hbar_{3,0}\varkappa_0\mathcal{S}_0''\right],\end{aligned}\tag{S23}$$

where $(\Bbbk_i \varkappa)'_0 \equiv \partial_q (\Bbbk_i \varkappa)\big|_{q=q_0}$. The corresponding eigenfrequencies of domain wall osculations in vicinity of the equilibrium for different geometries as function of cross section deformation are presented in Fig. 3.

SM-4. EFFECTIVE MODEL OF CURVED BIAXIAL STRIPE

In previous sections we discussed the role of varying cross section by separating effects of curvature and torsion from effects connected with cross section. Another approach to the same problem is to reduce the model to that of a curved stripe with a fixed cross section, but with curvature, torsion and local anisotropy modified by the varying cross section.

We start from the same model of curved magnet with varying cross section (1). By applying the transformation $\zeta(\xi) = \int \mathrm{d}\xi/\mathcal{S}(\xi)$, the total energy $\mathscr{E} = E/E_0 = \int \mathscr{E}^\star(\zeta)\mathrm{d}\zeta$ resembles the energy of curved biaxial magnet with constant cross section:

$$\begin{aligned}&\mathscr{E}^\star = \mathscr{E}_0 + \underbrace{\mathscr{E}_{\text{A}}}_{\substack{\text{effective}\\ \text{anisotropy}}} + \underbrace{\mathscr{E}_{\text{D}}}_{\substack{\text{effective}\\ \text{DMI}}}, \quad \mathscr{E}_0 = \partial_\zeta m_i \partial_\zeta m_i,\\ &\mathscr{E}_{\text{A}} = \mathscr{K}^\star_{ij} m_i m_j, \quad \mathscr{E}_{\text{D}} = \varepsilon_{ijk}\mathscr{D}^\star_i m_j \partial_\zeta m_j,\end{aligned} \tag{S24}$$

where $\mathscr{K}^\star_{ij}(\zeta) = \mathcal{S}^2(\zeta)\mathscr{K}_{ij}$ and $\mathscr{D}^\star_i = \mathcal{S}(\zeta)\mathscr{D}_i$ are effective coordinate dependent coefficients of anisotropy and DMI, respectively. Model (S24) corresponds to a chiral biaxial ferromagnet with coordinate-dependent anisotropy coefficients and DMI.

First, we will calculate an emergent geometry-induced magnetic field (2). In new variables this field is defined as

$$\begin{aligned}&\mathscr{E}^\star \approx \mathscr{E}_{\text{T}} - \boldsymbol{F}^\star \cdot \boldsymbol{m} + k^\star_{\text{T}}\left(\vartheta^2 + \varphi^2\right) + k^\star_{\text{B}}\vartheta,\\ &\boldsymbol{F}^\star = 2\mathfrak{C}\partial_\zeta \varkappa^\star \boldsymbol{e}_{\text{N}} + 2\mathfrak{C}\varkappa^\star\sigma^\star \boldsymbol{e}_{\text{B}},\end{aligned} \tag{S25}$$

where $\varkappa^\star = \mathcal{S}(\zeta)\varkappa$, $\sigma^\star = \mathcal{S}(\zeta)\sigma$, $k^\star_{\text{T}} = \mathcal{S}^2(\zeta)k_{\text{T}}$, and $k^\star_{\text{B}} = \mathcal{S}^2(\zeta)k_{\text{B}}$ are redefined parameters of the system. As a result, the magnetization tilting is defined as

$$\vartheta \approx -\frac{\mathfrak{C}}{k^\star_{\text{T}} + k^\star_{\text{B}}}\varkappa^\star\sigma^\star, \quad \varphi \approx \frac{\partial_\zeta \varkappa^\star}{k^\star_{\text{T}}}. \tag{S26}$$

A. Domain wall properties

As a next step, we consider domain wall properties within model (S24). Using curvilinear reference frame, one can parameterize the magnetization as $\boldsymbol{m} = \boldsymbol{e}_{\text{T}}\cos\theta + \boldsymbol{e}_{\text{N}}\sin\theta\cos\phi + \boldsymbol{e}_{\text{B}}\sin\theta\sin\phi$, where $\theta = \theta(\zeta)$ and $\phi = \phi(\zeta)$ are magnetic angles. The minimization of the energy results in a planar texture within the stripe plane with $\cos\phi = \mathscr{C} = \pm 1$ and planar deviations from the tangential direction described by $\theta(\zeta)$, which satisfies the driven nonlinear pendulum equation

$$\partial_{\zeta\zeta}\theta - k^\star_{\text{T}}\cos\theta\sin\theta = f^\star(\zeta), \quad f^\star(\zeta) = -\mathscr{C}\partial_\zeta\varkappa^\star. \tag{S27}$$

The spatially dependent external force $f^\star(\zeta)$ results in the absence of strictly tangential magnetization pattern.

In order to analyze domain wall properties we apply a collective variable approach based on the q-Φ model [16, 17]

$$\cos\theta^{\text{DW}} = -p\tanh\left[\frac{\zeta - q(t)}{\Delta}\right], \quad \phi^{\text{DW}} = \Phi(t). \tag{S28}$$

When the force is absent, $f^\star(\zeta) = 0$, this model provides an exact solution of (S27); it describes head-to-head or tail-to-tail domain walls with domain wall width $\Delta = 1/\sqrt{k^\star_{\text{T}}}$ for $p = 1$ and $p = -1$, respectively.

By substituting Ansatz (S28) into the energy density (S24) and performing integration over the ζ we get an effective domain wall energy (for the case of narrow domain wall)

$$\begin{aligned}\frac{\mathscr{E}^{\text{DW}}_\star}{2} \approx &\frac{1}{\Delta} + k^\star_{\text{T}}(q)\Delta + p\pi\varkappa^\star(q)\cos\Phi\\ &+ \left\{k^\star_{\text{B}} - \left[\varkappa^\star(q)\right]^2\right\}\sin^2\Phi.\end{aligned} \tag{S29}$$

Equilibrium values of the domain wall position q_0, its phase Φ_0, and the domain wall width Δ_0 can be found by the energy minimization. The corresponding values are determined by the equations

$$\begin{aligned}&\frac{\left(\partial_\zeta k^\star_{\text{T}}\right)_0}{\sqrt{k^\star_{\text{T},0}}} = \pi\,\mathrm{sgn}\,\varkappa^\star_0\left(\partial_\zeta\varkappa^\star\right)_0,\\ &\Delta_0 = \frac{1}{\sqrt{k^\star_{\text{T},0}}}, \quad \cos\Phi_0 = -p\,\mathrm{sgn}\,\varkappa^\star_0.\end{aligned} \tag{S30}$$

where $\left(\partial_\zeta g^\star\right)_0 \equiv \partial_\zeta g^\star_{\text{T}}\big|_{q=q_0}$ and $k^\star_{\text{T},0} = k^\star_{\text{T}}(q_0)$.

In the same manner we construct the effective Lagrangian of the domain wall $\mathscr{L}^{\text{DW}}_\star = \mathscr{L}\left[\theta^{\text{DW}}, \phi^{\text{DW}}\right]$ and effective dissipative function $\mathscr{R}^{\text{DW}}_\star = \mathscr{R}\left[\theta^{\text{DW}}, \phi^{\text{DW}}\right]$:

$$\begin{aligned}&\mathscr{L}^{\text{DW}}_\star = 2p\Phi\dot{q} - \mathscr{E}^{\text{DW}}_\star,\\ &\mathscr{R}^{\text{DW}}_\star = \frac{\alpha}{\Delta}\left(\dot{q}^2 + \Delta^2\dot{\Phi}^2\right).\end{aligned} \tag{S31}$$

By substituting the effective Lagrangian and the dissipative function into the Lagrange–Rayleigh equations (S17), one gets effective equations of motion of the narrow wall:

$$\begin{aligned}\alpha\frac{\dot{q}}{\Delta_0} + p\dot{\Phi} = &\left[2\varkappa^\star(q)\Delta\sin^2\Phi - p\pi\cos\Phi\right]\partial_q\varkappa^\star(q)\\ &- \Delta\left[\partial_q k^\star_{\text{T}}(q) + \sin^2\Phi\,\partial_q k^\star_{\text{B}}(q)\right],\\ p\frac{\dot{q}}{\Delta_0} - \alpha\dot{\Phi} = &\left\{k^\star_{\text{B}} - \left[\varkappa^\star(q)\right]^2\right\}\sin 2\Phi - p\pi\frac{\varkappa^\star(q)}{\Delta_0}\sin\Phi.\end{aligned} \tag{S32}$$

Equations of motion (S32) linearized within the equilibrium state (S30) of domain wall reads as

$$\begin{aligned}&\begin{bmatrix}\dot{q}\\ \dot{\varphi}\end{bmatrix} \approx -\begin{bmatrix}\alpha\Omega^\star_{\text{g}} & -p\Delta_0\Omega^\star_{\text{A}}\\ p\Omega^\star_{\text{g}}/\Delta_0 & \alpha\Omega^\star_{\text{A}}\end{bmatrix}\cdot\begin{bmatrix}q\\ \varphi\end{bmatrix},\\ &\Omega^\star_{\text{g}} = \frac{\left(k^\star_{\text{T}}\right)''_0}{k^\star_{\text{T},0}} - \pi\Delta_0\,\mathrm{sgn}\,\varkappa^\star_0\left(\varkappa^\star\right)''_0,\\ &\Omega^\star_{\text{A}} = 2\left[k^\star_{\text{B}} - \left(\varkappa^\star_0\right)^2\right] + \frac{\pi}{\Delta_0}\left|\varkappa^\star_0\right|,\end{aligned} \tag{S33}$$

where $(g^\star)''_0 = \partial_{qq}(g^\star)\big|_{q=q_0}$.

Set of Eqs. (S33) results in a decaying oscillations $q(\bar{t}) = q_0 \sin\Omega\bar{t}\, e^{-\eta\bar{t}}$ and $\varphi(\bar{t}) = \varphi_0 \cos\Omega\bar{t}\, e^{-\eta\bar{t}}$ with frequency Ω and friction η defined as:

$$\Omega = \sqrt{\Omega^\star_{\text{A}}\Omega^\star_g}, \quad \eta = \frac{\alpha}{2}\left(\Omega^\star_{\text{A}} + \Omega^\star_g\right). \tag{S34}$$

SM-5. NUMERICAL SIMULATIONS

In order to verify our analytical results we perform numerical micromagnetic simulations of the Landau–Lifshitz–Gilbert equation utilizing the Nmag code [18]. We restrict ourselves to the case of magnetically soft material; therefore only two magnetic interactions are taken into account, namely exchange and magnetostatic contributions. In simulations we use material parameters of permalloy, namely, exchange constant $A = 2.6$ μerg/cm (in SI units $A^{\text{SI}} = 26$ pJ/m), saturation magnetization $M_s = 800$ G (in SI units $M_s^{\text{SI}} = 800$ kA/m). These parameters result in the exchange length $\ell \approx 5.7$ nm and $\omega_0 = 28.15$ GHz. Thermal effects and anisotropy are neglected. An irregular tetrahedral mesh with cell size about 2.75 nm is used.

In all simulations we considered stripes with total length $L = 500$ nm and cross section defined as $\mathcal{S}(\xi) = \mathfrak{s}_0\left\{1 - \varrho/\cosh\left[(\xi - \eta)/\lambda\right]\right\}$. Parameters $\mathfrak{s}_0 = 75/\ell^2$ and aspect ratio $a = 3$ are fixed for all simulations.

We consider three different geometries: (i) rectilinear stripe with $\varkappa = \varkappa' = 0$; (ii) circle segment with curvature $\varkappa_0 = 0.1$ and $\varkappa' = 0$ (for circle geometry we did not consider closed loop in order to have only one domain wall); (iii) the parabola geometry $\boldsymbol{\gamma} = x\hat{\boldsymbol{x}} + \varkappa_0 x^2 \hat{\boldsymbol{y}}/(2\ell)$ with $\varkappa_0 = 0.05$ and $\varkappa_0 = 0.1$ being the extreme curvature.

The numerical experiment consists of three steps. Initially, we relax the domain wall structure in an overdamped regime ($\alpha = 0.25$) in order to determine the equilibrium values of collective variables: position q_0 and phase Φ_0. The obtained results fully coincide with the prediction (S15). To determine the values of q and Φ we extract the curvilinear magnetization components $m_{\text{T}} = \boldsymbol{m}\cdot\boldsymbol{e}_{\text{T}}$, $m_{\text{N}} = \boldsymbol{m}\cdot\boldsymbol{e}_{\text{N}}$, and $m_{\text{B}} = \boldsymbol{m}\cdot\boldsymbol{e}_{\text{B}}$ from the simulation data, and apply fitting with the Ansatz (5). Namely, the position q is determined as a fitting parameter for the function $m_{\text{T}}(\xi) = -p\tanh[(\xi - q)/\delta]$, then the phase is determined from the equation $\tan\Phi = m_{\text{B}}(q)/m_{\text{N}}(q)$.

In the second step we slightly perturb the domain wall phase Φ by applying a weak magnetic field $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$ perpendicularly to the stripe plane, where $B_0 = 15$ mT. After the system relaxation in the applied field $\boldsymbol{B}$ the field is switched off and the magnetization dynamics is simulated for the natural value of the damping coefficient $\alpha = 0.01$ (the third step). Since the variables q and Φ are canonically conjugated, see Eqs. (S20), the perturbed dynamics of Φ induces the dynamics of q, i.e., the domain wall starts to move, for details see movies:

1. `parabola_stripe_eta_0.mp4` shows domain wall oscillations in a parabola shaped stripe with $\varrho = 0.9$, $\eta = 0$, and $\lambda = 15$ and maximal curvature $\varkappa_0 = 0.2$.

2. `parabola_stripe_eta_7.5.mp4` shows domain wall oscillations in a parabola shaped stripe with $\varrho = 0.9$, $\eta = 7.5$, and $\lambda = 15$ and maximal curvature $\varkappa_0 = 0.2$.

3. `circle_segment.mp4` shows domain wall oscillations in a circle segment shaped stripe with $\varrho \approx 0.67$, $\eta = 0$, $\lambda = 10$, and curvature $\varkappa_0 = 0.05$.

4. `straight_stripe.mp4` shows domain wall oscillations in a straight stripe with $\varrho = 0.9$, $\eta = 0$, and $\lambda = 15$.

In all cases Gilbert damping parameter was fixed $\alpha = 0.01$.

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