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Field momentum and gyroscopic dynamics of classical systems with topological defects

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Abstract
The standard relation between the field momentum and the force is generalized for the system with a field singularity: in addition to the regular force, there appears the singular one. This approach is applied to the description of the gyroscopic dynamics of the classical field with topological defects. The collective-variable Lagrangian description is considered for gyroscopical systems taking into account singularities. Using this method, we describe the dynamics of two-dimensional magnetic solitons. We establish a relation between the gyroscopic force and the singular one. An effective Lagrangian description is discussed for the magnetic soliton dynamics.

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1. Introduction

An important role in the modern physics of condense matter and in field theories is connected with the study of topological defects. Common examples are specific distributions of the order parameter such as dislocations, disclinations, vortices, monopoles, hedgehogs, boojums, etc. At present the topological classification of defects is almost done, but the dynamical theory of defects is far from completeness. In the continuum approach, defects are described by essentially nonlinear solutions of field equations like topological solitons. The classical field theory in its standard form is suitable only for analysis of fields, which can be described by regular functions, while the soliton profile can be singular. This leads to ambiguities in the energy–momentum tensor problem: the linear momentum is either not well defined or is not conserved. Typical examples for this long-standing paradox in the condense matter theory are the magnetism, where there is no well-defined energy–momentum tensor; the canonical definition for the field momentum fails for the magnetic bubbles [1]; this canonical momentum is not invariant under spin rotations [2]. The part of the problem, which is connected with the absence of the momentum invariance under gauge transformation, can be explained on the microscopic quantum level as a result of momentum exchange with microscopic degrees of freedom [3]. In this case, it is possible to treat the problem by introducing the nonlocal Novikov–Wess–Zumino term in the action [3, 4].

A new discussion of the momentum problem appeared in the last decade due to the study of dynamics of topological solitons in low-dimensional magnetism. In particular, the usage of canonical momentum for the construction of the effective equation of motion for magnetic vortices leads to contradictions between different approaches [5–8]. Let us note that the problem was solved by Papanicolau and Tomaras [6] for the special case of localized magnetic solitons (often named 'skyrmions'), where the nonstandard form of field momentum was constructed as a moment of vorticity; however, such approach is not universal.

In this paper, we show how to avoid the problem in terms of standardly defined field momentum. We construct the equation of motion, involving the force and the momentum, which is suitable for the description of singular objects like topological defects. By generalizing an equation for the energy–momentum flux, we calculate the relation between the time derivative of the momentum and the force acting on the system (Newtonian-like equation). We prove in section 2 that in addition to the regular force there appears the singular one, which exists in the system with the singular distribution of the field. This generalized approach works for a large class of models. We use this method in section 3 to describe the dynamics of gyroscopic systems. Our approach is applied to the problem of collective-variable Lagrangian description of gyroscopic systems; effective equation of motion are constructed in section 4. For gyroscopic systems in two-dimensional (2D) magnetism, we present in section 5 explicit results for different models. The connection between the gyroscopic force and the singular one is discussed. We consider the possibility of using the collective-variable Lagrangian approach in the magnetic solitons dynamics. We conclude in section 6.

2. Energy and force-balance equations

We study the classical Lagrangian dynamics for the multicomponent field $\mathbf{\Phi}(\boldsymbol{x},t)$ in the $(d+1)$ spacetime dimensions, which is described by Euler–Lagrange equations:

$$\frac{\delta L}{\delta \Phi_k} \equiv \frac{\partial \mathscr{L}}{\partial \Phi_k} - \partial_\alpha \frac{\partial \mathscr{L}}{\partial \Phi_{k,\alpha}} = 0, \tag{1}$$

where $L = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{x}\, \mathscr{L}(\Phi_k; \Phi_{k,\alpha})$. Here and below Latin indices k, l describe components of the field $\mathbf{\Phi}$, Latin indices $i, j = \overline{1, d}$ numerate spatial coordinates x_i and Greek indices $\alpha, \beta = \overline{0, d}$ correspond to spacetime coordinates.

We start with the standard definition of the field momentum

$$\boldsymbol{P} = -\int_{\mathcal{D}} \mathrm{d}\boldsymbol{x} \frac{\partial \mathscr{L}}{\partial \Phi_{k,0}} \nabla \Phi_k. \tag{2}$$

To describe the dynamics of the system as a whole on the basic of the momentum (2), let us consider the energy–momentum tensor [9]

$$T_{\alpha\beta} = \Phi_{k,\alpha} \frac{\partial \mathscr{L}}{\partial \Phi_{k,\beta}} - \mathscr{L}\delta_{\alpha\beta}. \tag{3}$$

The flux of the energy–momentum tensor can be calculated in the standard way:

$$\partial_\beta T_{\alpha\beta} = \Phi_{k,\alpha,\beta} \frac{\partial \mathscr{L}}{\partial \Phi_{k,\beta}} + \Phi_{k,\alpha} \partial_\beta \frac{\partial \mathscr{L}}{\partial \Phi_{k,\beta}} - \partial_\alpha \mathscr{L}. \tag{4}$$

Calculating the derivative $\partial_\alpha \mathscr{L}$ taking into account (1) and changing the order of the derivation as follows, $\Phi_{k,\alpha,\beta} = \Phi_{k,\beta,\alpha}$, we obtain the well-known equation [9]

$$\partial_\beta T_{\alpha\beta} = 0. \tag{5}$$

In the integral form equation (5) with $\alpha = 0$ corresponds to the work equation for the total energy $\mathcal{E} = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{x}\, T_{00}$,

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -\oint_{\partial\mathcal{D}} \mathrm{d}f_i \mathcal{S}_i, \tag{6}$$

which means that the energy changes due to the flux through the boundary, $\mathcal{S}_i = T_{0i}$. Space components of the integral equation following from (5) give the Newtonian-like equation

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \boldsymbol{F}^{\mathrm{reg}}, \qquad F_i^{\mathrm{reg}} = -\oint_{\partial\mathcal{D}} \mathrm{d}f_j \Pi_{ij}, \qquad \Pi_{ij} = \mathscr{L}\delta_{ij} - \Phi_{k,i}\frac{\partial\mathscr{L}}{\partial\Phi_{k,j}}. \tag{7}$$

Let us recall that to derive relations (5)–(7), it is necessary to suppose that second derivatives of any field components commute, $[\partial_\alpha, \partial_\beta]\Phi_k = 0$. This is a standard assumption, which works well for the smooth distribution of the field $\boldsymbol{\Phi}$. However, for the system with topological defects this assumption can fail.

Probably, the most familiar kind of singularity is the *phase singularity* [10], which can be found in different physical systems. In a light wave, the phase singularity is known as an optical vortex [11]; such a singular phenomenon gives birth to the singular optics. One of the well-known example of the phase singularity in the condense matter physics is the 2D quantum Hall systems [12], where the Chern–Simons approach is employed by making the singular gauge transformation on the phase of the electron wavefunction. In the simplest case of a single electron, this transformation can be written as $\psi \to \phi \cdot \psi$, where $\phi = \mathrm{e}^{-\mathrm{i}\arg(z-z_0)}$ and $z \in \mathbb{C}$ is a point in the xy-plane. This leads to the additional Chern–Simons magnetic field with the vector potential (or the Berry connection in accordance to [13]) $\boldsymbol{A} = \mathrm{i}\nabla\phi$, which has a singularity at $z = z_0$ due to the multivalued function $\arg(z)$. The corresponding magnetic induction does not vanish at z_0, namely $\boldsymbol{B} = \nabla \times \nabla\phi = 2\pi\delta(z - z_0)\boldsymbol{e}_3$, where $\delta(z)$ is 2D Dirac's delta function. The same situation takes place for the Aharonov–Bohm effect, where the Berry phase ϕ can be interpreted as an Aharonov–Bohm phase [13]. The singular field distribution appears for 2D solitons in magnets, where the in-plane angle of magnetization is described by the multivalued $\arg(z)$-function, $\phi = \arg(z - z_0)$ [14]. The second derivatives of ϕ do not commute, $\epsilon_{ij}\partial_i\partial_j\phi = 2\pi\delta(z - z_0)$, this is well discussed by Papanicolau and Tomaras [6]. All above-mentioned singularities are connected with Dirac's monopole: the vector potential has a Dirac string along some direction (in our case the string crosses an xy-plane at z_0), which breaks the invariance of the system [2].

Let us calculate the energy–momentum dynamics equations, which allow the field singularities. Simple calculations taking into account (1), (3) and (4) lead to *the generalized expression* for the energy–momentum flux:

$$\partial_\beta T_{\alpha\beta} = \frac{\partial\mathscr{L}}{\partial\Phi_{k,\beta}}(\Phi_{k,\alpha,\beta} - \Phi_{k,\beta,\alpha}). \tag{8}$$

In general case there exist a nonzero flux of the energy–momentum, so the conservation laws in the system can vanish. A similar picture, when the energy–momentum tensor cannot be presented in the covariant form, takes place in the general relativity [9]. In the fluid dynamics such a singularity is known for vortices [15]. Below we discuss several examples in the condense matter physics, in particular, in the magnetism, where such a singularity is connected to the gyroscopical dynamics of topological excitations.

Using (8) one can derive the work equation in the form

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -\oint_{\partial\mathcal{D}} \mathrm{d}f_i\, \mathcal{S}_i + \int_{\mathcal{D}} \mathrm{d}\boldsymbol{x} \frac{\partial\mathscr{L}}{\partial\Phi_{k,i}}(\Phi_{k,i,0} - \Phi_{k,0,i}). \tag{9}$$

The energy changes not only due to the flux through the boundary as in (6). The second term on the right-hand side (RHS) of the work equation (9) describes the energy changes due to field singularities.

The space components of the integral form of (8) can be presented in the Newtonian way, similar to (7),

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \boldsymbol{F}, \qquad \boldsymbol{F} = \boldsymbol{F}^{\mathrm{reg}} + \boldsymbol{F}^{\mathrm{sing}}. \tag{10a}$$

The force has two contributions: one of them, $\boldsymbol{F}^{\mathrm{reg}}$, can be expressed as the current of the stress tensor Π_{ij}, see (7). An additional *singular force*

$$F_i^{\mathrm{sing}} = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{x} \frac{\partial \mathscr{L}}{\partial \Phi_{k,\beta}} (\Phi_{k,\beta,i} - \Phi_{k,i,\beta}) \tag{10b}$$

appears only if the field distribution has a singularity (when derivatives of $\boldsymbol{\Phi}$ are not smooth in $\mathcal{D}$). Namely, this additional force $\boldsymbol{F}^{\mathrm{sing}}$ is the main issue of our investigation.

If the field distribution $\boldsymbol{\Phi}(\boldsymbol{x}, t)$ is calculated, then equation (10a) describes the effective equation of motion for the system as a whole. Such an approach is known to be applied to the dynamics of regular fields, where it takes the form (7), see [14, 16]. Existence of the force $\boldsymbol{F}^{\mathrm{sing}}$ is caused by the additional flux through the region of the field singularity.

Let us discuss the possible candidates who admit effects of $\boldsymbol{F}^{\mathrm{sing}}$. An explicit form of this force (10b) shows that it is absent for one-dimensional (1D) systems, where $[\partial_0, \partial_1]\Phi_k = 0$. That is why it is possible to use the standard force balance (7) for the description of the dynamics of 1D solitons [14].

Apparently, the singular force $\boldsymbol{F}^{\mathrm{sing}}$ can appear in systems, where the Lagrangians contain non-potential terms, because the energy density should be finite. Such properties have gyroscopical systems. Therefore, the generalized force-balance equation (10a) taking into account singular force $\boldsymbol{F}^{\mathrm{sing}}$ should be used for the description of the gyroscopic dynamics for systems with singular topological solitons. Note that the usage of the standard force balance in the form (7) leads to the discrepancy in the definition of the gyroscopic force between the soliton perturbation theory [5] and direct integration of the field equations [1, 17, 18].

3. Gyroscopic systems in the field theory

Let us consider the field system, whose dynamics has only gyroscopic properties. The Lagrangian of such a simplest gyroscopic system has the form

$$\mathscr{L}(\Phi_k; \Phi_{k,\alpha}) = \mathscr{G} - \mathscr{H} \equiv A_k(\boldsymbol{\Phi})\Phi_{k,0} - \mathscr{H}. \tag{11}$$

We suppose that the 'Hamiltonian' $\mathscr{H}$ is a regular function of $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}_{,i}$, and all peculiarities can appear only due to the gyroscopic term $\mathscr{G} = A_k\Phi_{k,0}$. Such a form of the Lagrangian corresponds to the case of a system with regular gyroscopic matrix, which was systematically studied in [19], using a collective-variable theory for constrained Hamiltonian systems of a classical mechanics.

The Euler–Lagrange equations for this system have the form

$$\mathcal{G}_{kl}\Phi_{l,0} = \frac{\delta H}{\delta \Phi_k} \tag{12}$$

with the antisymmetric gyroscopic tensor $\mathcal{G}_{kl} = \partial A_l/\partial\Phi_k - \partial A_k/\partial\Phi_l$.

Let us calculate integral Newtonian equations in the form (10a). The field momentum for the system (11) has a gyroscopical nature,

$$\boldsymbol{P}^{(g)} = -\int_{\mathcal{D}} \mathrm{d}\boldsymbol{x}\, A_k \nabla \Phi_k.$$

Let us start with a regular field distribution, when $\mathrm{d}\boldsymbol{P}^{(g)}/\mathrm{d}t = \boldsymbol{F}^{\mathrm{reg}}$, see (7). Supposing that the field distribution is also localized, one can write the Newtonian equation in the form of the force-balance condition:

$$\boldsymbol{F}^{(g)} + \boldsymbol{F}^{\mathrm{reg}(H)} = 0, \qquad F_i^{\mathrm{reg}(H)} = \oint_{\partial\mathcal{D}} \mathrm{d}f_j \left(\mathscr{H}\delta_{ij} - \Phi_{k,i}\frac{\partial \mathscr{H}}{\partial \Phi_{k,j}} \right). \tag{13}$$

Here, the quantity

$$\boldsymbol{F}^{(g)} = -\frac{\mathrm{d}\boldsymbol{P}^{(g)}}{\mathrm{d}t} \tag{14}$$

is an 'internal' gyroscopic force, which acts together with external force $\boldsymbol{F}^{\mathrm{reg}(H)}$ on the system. The gyroscopic force in this form was introduced in [20] and used after that for the description of regular field distributions, see for the review [16].

The picture drastically changes if we consider singular field distributions. Let us write the force-balance equation (10*a*), separating the gyroscopic contribution:

$$\frac{\mathrm{d}\boldsymbol{P}^{(g)}}{\mathrm{d}t} = \boldsymbol{F}^{\mathrm{reg}(g)} + \boldsymbol{F}^{\mathrm{reg}(H)} + \boldsymbol{F}^{\mathrm{sing}(g)}, \tag{15a}$$

$$F_i^{\mathrm{reg}(g)} = -\oint_{\partial\mathcal{D}} \mathrm{d}f_i A_k \Phi_{k,0}, \tag{15b}$$

$$F_i^{\mathrm{sing}(g)} = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{x} A_k (\Phi_{k,0,i} - \Phi_{k,i,0}). \tag{15c}$$

To fashion the Newtonian equation (15*a*) as the force-balance condition (13), we define the gyroscopic force as follows:

$$\boldsymbol{F}^{(g)} = -\frac{\mathrm{d}\boldsymbol{P}^{(g)}}{\mathrm{d}t} + \boldsymbol{F}^{\mathrm{reg}(g)} + \boldsymbol{F}^{\mathrm{sing}(g)}. \tag{16}$$

This definition of the gyroscopic force differs from the usual one (14). Note that using the gauge transformation $A_k \to A_k - A_k^{\mathrm{ground}}$ it is possible to suppress an effect of $\boldsymbol{F}^{\mathrm{reg}(g)}$. Nevertheless, the presence of the singular force (15*c*) breaks the simple relation (14). Moreover, we will see in equation (37) that for magnetic systems the gyroscopic force and the singular one have the same value, $|\boldsymbol{F}^{(g)}| = |\boldsymbol{F}^{\mathrm{sing}}|$.

One can rewrite the complicated expression (16) for the gyroscopic force in the compact form

$$F_i^{(g)} = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{x}\, \mathcal{G}_{kl} \Phi_{k,0} \Phi_{l,i}, \tag{17}$$

which can be used for describing both localized topological solitons (skyrmions) [16, 17, 20] and nonlocalized vortices [5, 7, 8].

It is easy to generalize results for systems, whose dynamics admit both kinetic and gyroscopic properties. Let us start with the Lagrangian system

$$\mathscr{L} = \mathscr{G} + \mathscr{L}^{(0)},$$

where we separate the gyroscopic term $\mathscr{G}$ from the Lagrangian and suppose that $\mathscr{L}^{(0)}$ has no singularities. The simple generalization of the force-balance relation takes a form

$$\frac{\mathrm{d}\boldsymbol{P}^{(0)}}{\mathrm{d}t} = \boldsymbol{F}^{(0)} + \boldsymbol{F}^{(g)}, \qquad F_i^{(0)} = -\oint_{\partial\mathcal{D}} \mathrm{d}f_j \left(\mathscr{L}^{(0)}\delta_{ij} - \Phi_{k,i}\frac{\partial \mathscr{L}^{(0)}}{\partial \Phi_{k,j}} \right). \tag{18}$$

It is instructive to mention an analogy with an equation of motions of the charged particle m in the electromagnetic field $\boldsymbol{A}$ under the action of the external force $\boldsymbol{F}$. Since the canonical momentum of the particle is $\boldsymbol{P} = m\boldsymbol{v} + \boldsymbol{A}$, the Newtonian equation of motion takes a form

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \boldsymbol{F} \qquad \text{or} \qquad m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{F} - \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}. \tag{19}$$

The last term $-\mathrm{d}\boldsymbol{A}/\mathrm{d}t$ can be interpreted as a Lorentz force, the particular case of a gyroscopic force, in analogy with the relation $\boldsymbol{F}^{(g)} = -\mathrm{d}\boldsymbol{P}^{(g)}/\mathrm{d}t$, see (14). Note that the sign 'minus' always appears in the gyroscopic force, because internal gyroscopical properties of the whole system (in the example (19) this system consists of the particle and the electromagnetic field) are interpreted as an additional force, which acts on a particle.

4. Thiele approach and effective Lagrangian

Let us consider the collective-variable dynamics of the gyroscopic system with the Lagrangian (11). The collective-variable description becomes important in the nonlinear field theories, when the field distribution has well-defined particle-like properties. If the system admits the travelling wave solution (*travelling wave ansatz*, TWA),

$$\Phi_k^{\mathrm{TWA}}(\boldsymbol{x}, t) = \Phi_k(\boldsymbol{x} - \boldsymbol{X}(t)), \tag{20}$$

one can derive the gyroscopic force (17) in the form

$$F_i^{(g)} = G_{ij}\dot{X}_j. \tag{21a}$$

Here, the gyroscopic tensor

$$G_{ij} = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{x}\, \mathcal{G}_{kl}\Phi_{k,i}\Phi_{l,j} \tag{21b}$$

is an extension of the gyrocoupling tensor, obtained by Thiele [17], to general gyroscopic systems. Then, the force-balance condition takes the form of Thiele-like equations, cf [17]:

$$G_{ij}\dot{X}_j + F_i(\boldsymbol{X}) = 0, \qquad \boldsymbol{F} \equiv \boldsymbol{F}^{\mathrm{reg}(H)} = -\frac{\partial H}{\partial \boldsymbol{X}}, \tag{22}$$

where $H = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{x}\, \mathscr{H}$. Note that one can derive Thiele-like equations from the effective Lagrangian:

$$L^{\mathrm{eff}} = \frac{1}{2}G_{ij}X_i\dot{X}_j - H, \qquad G_{ij} = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{x} \left(\frac{\partial A_l}{\partial \Phi_k} - \frac{\partial A_k}{\partial \Phi_l}\right)\Phi_{k,i}\Phi_{l,j}. \tag{23}$$

The generalization of Thiele-like equations (22) in the spirit of collective-variable theory can be made for the case when there is no exact travelling wave solution. The basis of this theory is a *generalized travelling wave ansatz*: [8, 21]

$$\Phi_k(\boldsymbol{x}, t) = \Phi_k\big(\boldsymbol{x} - \boldsymbol{X}(t), \partial_0\boldsymbol{X}(t), \partial_0^2\boldsymbol{X}(t), \ldots, \partial_0^n\boldsymbol{X}(t)\big),$$

which leads to the $(n + 1)$th-order equation of motion:

$$\sum_{k=1}^{n+1} G_{ij}^k \partial_0^k X_j + F_i(\boldsymbol{X}) = 0. \tag{24}$$

Note that in the Thiele approximation $n = 0$ and $G_{ij}^1 = G_{ij}$.

Another kind of a generalization appears when internal degrees of freedom become important. For example, in the Rice approach [22] for the 1D Klein–Gordon model the kink width becomes a collective variable as well as its position. Generalization for 2D solitons and vortices has been done recently in [23, 24]. That is why we will discuss here the possibility of deriving the effective Lagrangian of the system directly by integrating the microscopic Lagrangian (11) with the travelling wave ansatz (20).

Let us define the effective Lagrangian of the gyroscopic system:

$$L^{\text{eff}} = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{x}\, \mathscr{L}\big(\Phi_k^{\text{TWA}};\, \Phi_{k,\alpha}^{\text{TWA}}\big).$$

It is easy to see that the effective momentum coincides with the standard field momentum, calculated with the travelling wave ansatz, $\partial L^{\text{eff}}/\partial \dot{\boldsymbol{X}} = \boldsymbol{P}$. In the same way one can calculate the effective force $\partial L^{\text{eff}}/\partial \boldsymbol{X}$, which is equal to the regular force $\boldsymbol{F}^{\text{reg}}$. Thus effective Euler–Lagrange equations have the form of the singular force-balance condition (10), which can be presented as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^{\text{eff}}}{\partial \dot{\boldsymbol{X}}} - \frac{\partial L^{\text{eff}}}{\partial \boldsymbol{X}} = \boldsymbol{F}^{\text{sing}}, \qquad F_i^{\text{sing}} = \dot{X}_j \int_{\mathcal{D}} \mathrm{d}\boldsymbol{x}\, A_k(\Phi_{k,i,j} - \Phi_{k,j,i}). \tag{25}$$

The standard effective Lagrangian description is adequate *only* when the singular force is absent.

Let us consider the situation when we should obviate difficulties with the singular force. The gauge transformation $A_k \to A_k + \partial f(\boldsymbol{\Phi})/\partial \Phi_k$ changes the gyroscopic tensor by the value

$$\mathcal{G}_{kl}^{\text{gauge}} = \frac{\partial^2 f}{\partial \Phi_k \partial \Phi_l} - \frac{\partial^2 f}{\partial \Phi_k \partial \Phi_l}.$$

If the function $f(\boldsymbol{\Phi})$ is smooth enough and the second derivatives commute, the gauge transformation does not change equations of motions (12). Nevertheless, there could appear uncertainty in the canonic momentum definition. Under the gauge transformation, the momentum changes by the value

$$\boldsymbol{P}^{\text{gauge}} = -\int_{\mathcal{D}} \mathrm{d}\boldsymbol{x}\, \frac{\partial f}{\partial \Phi_k} \boldsymbol{\nabla} \Phi_k,$$

which is not well defined for the singular field distributions [2, 4]. If $A_k(\boldsymbol{\Phi}(\boldsymbol{x}))$ takes the value $A_k^{\text{sing}} = A_k(\boldsymbol{\Phi}(\boldsymbol{x}_0))$ in a singular point $\boldsymbol{x}_0$ of the field $\boldsymbol{\Phi}$, then after the gauge transformation $A_k \to A_k - A_k^{\text{sing}}$ the Lagrangian (11) will have no singularity. Thus, the following effective Lagrangian approach is valid:

$$L^{\text{eff}} = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{x}\big[\big(A_k - A_k^{\text{sing}}\big)\Phi_{k,0}^{\text{TWA}} - \mathscr{H}\big], \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^{\text{eff}}}{\partial \dot{\boldsymbol{X}}} - \frac{\partial L^{\text{eff}}}{\partial \boldsymbol{X}} = 0. \tag{26}$$

Such an approach can be generalized for the case when the field $\boldsymbol{\Phi}$ has several singular points $\boldsymbol{x}_n$, but with the same behaviour, $A_k^{\text{sing}} = A_k(\boldsymbol{\Phi}(\boldsymbol{x}_n))$. To illustrate this effective Lagrangian method (26) we construct below an effective Lagrangian for the magnetic vortex dynamics, see (40).

5. Application to the 2D magnetism

In this section, we apply our results to the dynamical properties of 2D topological defects (solitons and vortices) in magnetic systems. In the continuum limit, the dynamics of the broad class of Heisenberg magnets can be described in terms of the unit order parameter vector $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$; for the classical ferromagnet $\boldsymbol{n}$ is the normalized magnetization, for the antiferromagnet $\boldsymbol{n}$ is the normalized sublattice magnetization vector. Thus, the magnet can be described by the two-component field $\boldsymbol{\Phi} = (\theta, \phi)$.

Let us start with the case of the ferromagnet, whose dynamics is described by Landau–Lifshitz equations [14]. Using $\pi \equiv \cos\theta$ as a canonical momentum for the azimuthal angle ϕ, dynamical equations take the form

$$\dot{\phi} = \frac{\delta H}{\delta \pi}, \qquad \dot{\pi} = -\frac{\delta H}{\delta \phi}. \tag{27}$$

Note that in spite of the fact that π and ϕ do have a form of a canonic pair, these variables are not well defined [4]: the azimuthal angle ϕ is ill defined when $\theta = 0, \pi$.

Topological properties of solutions are determined by the mapping of the xy-plane to the S^2-sphere of the order parameter space. This mapping is described by the homotopic group $\pi_2(S^2) = \mathbb{Z}$, which is characterized by the topological invariant (Pontryagin index):

$$Q = \frac{1}{4\pi}\int \mathrm{d}^2x\, \mathcal{Q}, \qquad \mathcal{Q} = \epsilon_{ij}\pi_{,i}\phi_{,j}. \tag{28}$$

The Pontryagin index takes integer values, $Q \in \mathbb{Z}$, being an integral of motion.

In the Lagrangian approach, one can derive Landau–Lifshitz equations (27) from the functional

$$L = -\int \mathrm{d}^2x (C - \cos\theta)\partial_0\phi - H, \tag{29}$$

where C is an arbitrary constant [5]. Usually, one chooses $C = 1$ in order to neglect the contribution of the ground state (which corresponds to $\theta = 0$ for easy-axis magnets) [14]. Then, the standard definition of the ferromagnet momentum integral reads

$$\boldsymbol{P} = \int \mathrm{d}^2x (1 - \cos\theta)\nabla\phi. \tag{30}$$

Note that namely this definition of the momentum is the origin of the long-time discussion in the literature [2, 3, 6–8]. The main criticism of the momentum definition (30) is connected with its conservation. It was shown by Haldane [2] that the momentum (30) cannot be conserved for a singular field distribution. The origin is in the singularity of the Lagrangian at some point. In order to visualize the singularity, let us rewrite the Lagrangian (29) using the dimension quantity $\boldsymbol{M} = M\boldsymbol{n}$ without constrain $\boldsymbol{M}^2 = \text{const}$:

$$\mathcal{L} = \boldsymbol{A}\cdot\partial_0\boldsymbol{M} - \mathcal{H}, \qquad \boldsymbol{A} = \frac{[\boldsymbol{n}_0 \times \boldsymbol{M}]}{M(M + \boldsymbol{n}_0\cdot\boldsymbol{M})}. \tag{31}$$

Here, $\boldsymbol{A}$ is the vector potential of effective magnetic field [25]. One can see that $\boldsymbol{A}$ has a singularity along the line $\boldsymbol{n}_0 \cdot \boldsymbol{M} = -M$. It is easy to calculate the magnetic induction of the 'magnetic field', $\boldsymbol{B} = \nabla_{\boldsymbol{M}} \times \boldsymbol{A} = -\boldsymbol{M}/M^3$, which coincides with a magnetic induction of a Dirac magnetic monopole. Thus, the vector potential $\boldsymbol{A}$ has a Dirac string along the direction $\boldsymbol{n}_0$, which breaks the rotation invariance of the model [2].

Since the Lagrangian (29) has a singularity, the standard momentum (30) is not well defined [2, 3], moreover it is not conserved. That is why Papanicolau and Tomaras [6] proposed another definition of the momentum, which is connected only with the topological properties of the ferromagnet (28):

$$P_i^{\mathrm{PT}} = \epsilon_{ij}\int \mathrm{d}^2x\, x_j \mathcal{Q}. \tag{32}$$

The momentum (32) is an analogue of a fluid impulse, which is defined as a linear moment of a local vorticity and used for the description of the fluid vortex dynamics [15].

The Poisson bracket relation for the momentum (32) takes the nonzero value, $\left\{P_1^{\mathrm{PT}}, P_2^{\mathrm{PT}}\right\} = 4\pi Q$ as well as for the standard momentum (30), $\{P_1, P_2\} = 4\pi Q$. An advantage of the momentum definition is its conservation for the finite energy field distribution [6].

The momentum $\boldsymbol{P}^{\mathrm{PT}}$ is claimed in [6] to be a generator of space translations. However, as it was shown in [7], the Poisson bracket between $\boldsymbol{P}^{\mathrm{PT}}$ and any smooth functional $F[\phi(\boldsymbol{x}), \pi(\boldsymbol{x})]$ takes the form

$$\left\{P_i^{\mathrm{PT}}, F\right\} = -\int \mathrm{d}^2x \left(\phi_{,i}\frac{\delta F}{\delta\phi} + \pi_{,i}\frac{\delta F}{\delta\pi}\right) + \epsilon_{ij}\epsilon_{kl}\int \mathrm{d}^2x\, x_j\phi_{,k,l}\frac{\delta F}{\delta\phi}. \tag{33}$$

Thus, $\boldsymbol{P}^{\mathrm{PT}}$ defines a true momentum functional only if the last term in (33) vanishes. It seems to vanish due to the antisymmetric tensor ϵ_{kl} is constrained with the symmetric $\phi_{,k,l}$. However, it is valid only for the regular field distribution. As an example let us consider the following singular distribution of the field, which corresponds to the simplest 2D topological defect:

$$\theta = \theta(|z - z_0|), \qquad \phi = Q\cdot\arg(z - z_0), \tag{34}$$

where $z = x + \mathrm{i}y$ and $z_0 \in \mathbb{C}$ is the position of the centre of the defect. The singular properties appear for the field variable ϕ: the second derivatives do not commute, $\epsilon_{kl}\phi_{,k,l} = 2\pi Q\delta(z-z_0)$. The last term in (33) finally reads $2\pi Q\epsilon_{ij}x_j\delta F/\delta\phi|_{z=z_0}$. One can see that in general the momentum $\boldsymbol{P}^{\mathrm{PT}}$ cannot be the translation generator.

Also note that the momentum $\boldsymbol{P}^{\mathrm{PT}}$ does not describe an individual soliton dynamics. Using the algebra for the momentum $\boldsymbol{P}^{\mathrm{PT}}$, it was shown in [6] that a single topological structure cannot move in the absence of an external field: the soliton with $Q \neq 0$ is always pinned at some point in xy-plane; it is possible to move the set of solitons [26]. This does not prevent the rotation motion of the soliton; however, the centre of the soliton orbit is fixed [6]. In this context, let us mention an analogy with the cyclotron motion of the electron: the electron coordinate changes when it moves along the cyclotron orbit, its standard momentum also changes, while their combination, the guiding centre position, is conserved. Namely, this guiding centre coordinate corresponds to the momentum $\boldsymbol{P}^{\mathrm{PT}}$, see [6].

One can see that the momentum $\boldsymbol{P}^{\mathrm{PT}}$ cannot provide an information about an instant soliton position, while the standard definition of the momentum gives a possibility of describing a single soliton motion, because it determines the gyroscopic force (16) and depends on the instant soliton position $\boldsymbol{X}(t)$, see (21a). The possibility of a single soliton motion was predicted in [27] for the easy-axis ferromagnet: it results from the complicated internal structure of the soliton. This motion was observed in simulations recently [28] by exciting a certain magnon mode, localized on the soliton.

Therefore, we go back to the standard definition of the ferromagnet momentum (30). Let us start with the localized distribution of the magnetization field, which corresponds to the magnetic skyrmion. We discuss here the problem of the conservation of the momentum (30), when an external force is absent, $\boldsymbol{F}^{\mathrm{reg}} = 0$. Using the force-balance equations (15), one can derive

$$\frac{\mathrm{d}P_i^{(g)}}{\mathrm{d}t} = F_i^{\mathrm{sing}} = \int_{\mathcal{D}} \mathrm{d}^2x(\cos\theta - 1)(\phi_{,0,i} - \phi_{,i,0}). \tag{35}$$

The total momentum is conserved only if $\boldsymbol{F}^{\mathrm{sing}} = 0$. However, the singular force does not vanish even for the simplest case of the Thiele-like motion of the soliton, which has a structure (34)

$$F_i^{\mathrm{sing}} = \epsilon_{ij}\Gamma\dot{X}_j, \qquad \Gamma = -4\pi Q,$$

where we suppose that in the centre of the soliton $\cos\theta = -1$. Let us calculate the gyroscopic force $\boldsymbol{F}^{(g)}$, which acts on the soliton from the media. Using (21), one can present the gyroscopic force in the form $F_i^{(g)} = \epsilon_{ij}G\dot{X}_j$, where the gyroscopic constant $G = 4\pi Q$. Thus, the gyroscopic force is caused by the field singularity, $G = -\Gamma$.

Table 1. Gyroscopic coefficients for different magnets: (1) easy axis (EA) and isotropic ferromagnet (FM) with spin S and lattice constant a [14]; (2) easy-plane (EP) FM in the perpendicular magnetic field $h = H/H_a$, parameter $p \equiv \cos\theta(0) = \pm 1$ describes the polarity of the vortex [5]; (3) EP antiferromagnet (AFM) in the perpendicular field H [29].

Type of magnet	Type of defect	$A(\theta)$	G
(1) EA FM	Solitons	$\hbar S a^{-2}(1-\cos\theta)$	$-4\pi Q\hbar S/a^2$
(2) EP FM	Vortex	$\hbar S a^{-2}\cdot(h-\cos\theta)$	$-2\pi Q(p-h)\hbar S/a^2$
(3) EP AFM	Vortex	$-(gH/c^2)\cdot\cos^2\theta$	$2\pi Q\cdot(gH/c^2)$

We have considered the case of magnetic skyrmions. It is possible to generalize results for different 2D topological defects. Let us consider the case of uniaxial 2D magnets, whose gyroscopic properties can be described by the following gyroscopic term in the Lagrangian: $\mathscr{G} = A(\theta)\partial_0\phi$. The form of the function $A(\theta)$ depends on the magnet type, see table 1, where we mention only models, which admit gyroscopic effects.

The simplest static nonlinear excitation in 2D magnets is the soliton for the isotropic and easy-axis magnets and the vortex for the easy-plane magnet. The structure of these different topological defects can be described by the field distributions (34). For standard models of the Heisenberg magnet all spatial derivatives $\partial\mathscr{L}/\partial\phi_{,i}$ vanish in the singularity point z_0, which is the centre of the defect; it agrees with arguments that the energy density should be finite. Therefore only the time derivative can influence the picture. The singular force takes the form, cf (15*c*),

$$F_i^{\text{sing}} = \int_{\mathcal{D}} \mathrm{d}^2x\, A(\theta)(\phi_{,0,i} - \phi_{,i,0}). \tag{36}$$

For the steady-state Thiele-like motion (20) this singular force is

$$F_i^{\text{sing}} = \epsilon_{ij}\Gamma\dot{X}_j, \qquad \Gamma = 2\pi Q A(z\to z_0).$$

One can see that $\boldsymbol{F}^{\text{sing}}$ has a gyroscopical behaviour. The gyroscopic force (21*a*) is determined by the gyroscopic tensor (21*b*):

$$F_i^{(g)} = \epsilon_{ij} G\dot{X}_j, \qquad G = 2\pi Q[A(z\to\infty) - A(z\to z_0)].$$

On the first view the gyroscopic constant is determined by the topological properties only: $G = -4\pi Q\cdot\hbar S/a^2$ for the soliton in the isotropic magnet and $G = -2\pi p Q\cdot\hbar S/a^2$ for the vortex in the easy-plane magnet, see table 1. However, if we switch on an external magnetic field, the gyroscopic constant G becomes a smooth function of the magnetic field, namely $G \propto (p-h)$ in the case of the cone-state ferromagnet and $G \propto H$ in the case of the antiferromagnet, see table 1. In general, the gyroscopic force is determined not only by the field distribution in the origin of the topological singularity, but also by the field distribution far from that. However, one can normalize the quantity A by the ground value, $A \to A - A(z\to\infty)$. Finally, the gyroscopic force reads

$$\boldsymbol{F}^{(g)} = -\boldsymbol{F}^{\text{sing}}. \tag{37}$$

This is an important relation between the gyroscopic force and the singular force, which assists to avoid the discrepancy between different approaches in the study of the gyroscopic properties of 2D magnetic solitons and vortices [5–8].

Let us discuss the possibility of using the collective-variable Lagrangian approach in the magnetic vortex dynamics. Usually, the Lagrangian of the ferromagnet is taken in the form (29) with $C = \cos\theta(\infty)$. For the easy-plane magnet $C = 0$ and $L = G - H$ with the gyroscopic term

$$G = \int \mathrm{d}^2x \cos\theta\, \partial_0\phi. \tag{38}$$

The field momentum $\boldsymbol{P} = -\int \mathrm{d}^2x \cos\theta \nabla\phi$ is commonly used in the magnetic vortex dynamics [8]. Let us consider the vortex in the circular 2D magnet of the radius L. It is well known [8] that the vortex in such a system rotates about the system centre due to the competition between the gyroscopic force and the image force, which imitates the interaction with a boundary. To describe the vortex motion in the Thiele approach, one needs to elaborate the model with the travelling wave ansatz (20). Simple calculations show that the gyroscopic term (38) disappears after the integration [24][1], and the effective Lagrangian *does not* contain the gyroscopic term, $L = -H$. Thus, Euler–Lagrangian equations cannot provide the well-known vortex rotation. The reason is the influence of the singular force: the Euler–Lagrangian equation for the effective Lagrangian contains an extra term $\boldsymbol{F}^{\text{sing}}$, see (25). We have derived this force above, it is opposite to the gyroscopic force, see (37).

In some cases it is possible to suppress the singularity and to construct the Lagrangian directly by integrating the field Lagrangian. We can do it formally as described in (26); but in order to visualize the singularity, let us consider the gauge transformation '$\cos\theta \to \cos\theta + \text{const}$' in the model (38). Under this transformation, the Lagrangian changes by the value

$$L^{\text{gauge}} = \text{const} \int_{\mathcal{D}} \mathrm{d}^2x\; \partial_0\phi, \tag{39}$$

which should not influence the Euler–Lagrange equations. However, the function ϕ is not differentiable, and this integral does not vanish: one can derive (39) using the travelling wave ansatz (20). After integrating, we obtain the gauge term in the form $L^{\text{gauge}} = \text{const}\, \pi Q \epsilon_{ij} X_i \dot{X}_j$, cf [24]. Thus, using the singular gauge transformation it is possible to suppress the singular force effect. In the case of the magnetic vortex with polarity p (see table 1), one can choose the regular effective Lagrangian

$$L^{\text{eff}} = \int_{\mathcal{D}} \mathrm{d}^2x \{[\cos\theta^{\text{TWA}} - p]\partial_0\phi^{\text{TWA}} - \mathscr{H}\}. \tag{40}$$

Generally, the regularized gyroscopic term for the 2D magnetic system can be presented in the form, cf, (26)

$$\mathscr{G} = [A(\theta^{\text{TWA}}(z)) - A(\theta^{\text{TWA}}(z_0))]\partial_0\phi^{\text{TWA}}.$$

We should note that such a simple picture works well when all singularities have the same behaviour, i.e. when θ-field takes the same value at all singular points. This can fail, e.g., for the system of two opposite polarized vortices [5]. In this case it is necessary to take into account singular force effects in the form of (25).

6. Conclusion

In conclusion, we have constructed the equation of motion, involving the force and the momentum, which is suitable for the description of singular objects like topological defects. This equation is a consequence of a more general problem of a noncovariance of the energy–momentum tensor (8). One of the well-known example of such a problem is the general relativity, when the energy–momentum tensor for the gravitational field cannot be presented in

[1] See equation (B7) in [24], where the gyroscopic term $\mathcal{G}_2$ corresponds to G in our equation (38).

the covariant form [9]. Another example is the fluid dynamics, where the Lagrangian principle is violated in the effective theory [15]. In the condense matter physics, in particular, in the magnetism, this nonlocality leads to the problem in the momentum definition. The reason of such paradoxes comes from the fact that the description of the many-body system in terms of few fields is always approximate, see the discussion in ([4] chapter 6). In the paper we did not describe the microscopic theory; however, we have shown how and when it is possible to resolve the problem using the energy–momentum tensor in the framework of the generalized expression (8) for the energy–momentum flux. We prove that in addition to the regular force there appears the singular one (10*b*). Effects of such a force are important for gyroscopical systems. We considered the gyroscopic dynamics of the classical field with topological defects and established relation (16) between the gyroscopic force, singular force and the time derivative of a standard field momentum. We have applied our approach to describe the gyroscopic properties of 2D topological defects (soliton and vortices) in 2D magnets and presented explicit results for different models. An important relation (37) is established between the gyroscopic force and the singular one: it shows that the gyroscopic properties are caused by the field singularity, which avoids contradictions between different approaches [5–8]. Using the singular force effects we also discuss the possibility of effective Lagrangian description, using collective coordinates' approach with an application for magnetic soliton and vortex dynamics in 2D magnets.

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