Dynamics of vortex ensemble in high-anisotropy planar ferromagnets with spin $s=1$

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The dynamics of an intraplanar vortex in a strongly anisotropic planar ferromagnet (PFM) with spin $s=1$ and strong effects of quantum spin contraction in an applied magnetic field $\mathbf{H}$ perpendicular to the easy plane is considered. The structure, dynamics, and relaxation of such vortices are discussed. Effective equations of motion are constructed for the coordinates of vortices in the ensemble of interacting vortices existing in the PFM above the topological phase transition point. The root-mean-square velocity of vortices is calculated as well as the components of the PFM dynamic structural factor in the presence of a magnetic field. © 1995 American Institute of Physics.

INTRAPLANAR VORTICES IN FERROMAGNETS

It is well known that topological solitons of the magnetic vortex type in two-dimensional (2D) magnets play the role of elementary excitations. Vortices are responsible for the violation of quasi-long-range order (the Berezinskii–Kosterlitz–Thouless transition)\cite{1,2} and lead to the emergence of a central peak (CP) in correlation functions.\cite{3-15} Central-peak effects were observed experimentally in the magnets BaNi$_2$(PO$_4$)$_2$,\cite{6} Rb$_2$CrCl$_4$,\cite{7} and CoCl$_2$,\cite{8} and described theoretically using analytical methods as well as numerical simulation. It should be noted that the value of energy of easy-plane anisotropy obtained in actual and numerical experiments was comparable with the exchange-interaction energy, while analytic descriptions were based on vortex solutions obtained for weakly anisotropic Heisenberg magnets using the Landau–Lifshitz equation.

For all values of the anisotropy constant for magnetic vortices, the magnetization at points separated by large distances from the center of a vortex lies in the easy plane ($xy$-plane) and is defined as

$$M_z = M_0 \cos \varphi, \quad M_y = M_0 \sin \varphi, \quad \varphi = \varphi_0 + \nu \chi, \quad r \to \infty,$$

where $M_0$ corresponds to the nominal (equilibrium) value of magnetization, $\varphi_0 = \text{const}; r$ and $\chi$ are the polar coordinates in the plane of the magnet, and the integer $\nu$ determines the polychoral charge of the vortex. (We assume that the center of the vortex coincides with the origin of the polar system of coordinates.) Thus, the magnetization turns through the angle $\nu \pi$ in the easy plane as we circumvent the vortex along a closed contour. A simple analysis\cite{16} shows that in the case of Heisenberg ferromagnet ($M_x^2 + M_y^2 + M_z^2 = M_0^2 = \text{const}$), vector $\mathbf{M}$ at the center of vortex must be orthogonal to easy plane. The dynamics of such (extraplanar) vortices is considered by Nikiforov and Sonin.\cite{17} It was shown that a vortex can move only if the magnet is characterized by a constant value of $\nabla \varphi$, and the vortex velocity $v \prod \varphi$. This property (i.e., freezing of a vortex in the condensed state according to the terminology used in Ref. 17) determines the basic features of the dynamics of a vortex ensemble like the value of the rms vortex velocity.\cite{3}

As the energy of anisotropy increases and the magnetization leaves the easy plane, the vortex obviously becomes less and less advantageous from the energy point of view. Taking into account the continuity of magnetization, we find that in the alternative version the length of magnetization at the center of the vortex ($r=0$) must vanish (see below). Thus, a description of intraplanar vortices based on a model in which magnetization is regarded as a continuous function of coordinates must take into account the variation of magnetization over the length.

Extraplanar as well as intraplanar vortices were observed in numerical experiments.\cite{10-15,18} It was found that extraplanar vortices are stable only in the case of a weak anisotropy, when $\lambda \in (\lambda_c, 1)$, while intraplanar vortices are stable for $\lambda \in (0, \lambda_c)$, where $\lambda$ characterizes the magnetic anisotropy, the quantity $1-\lambda$ determines the ratio of the anisotropy constant to the exchange integral, and $\lambda_c$ is the critical value of this constant, which depends on the shape of the magnetic lattice.\cite{10,18} According to recent results obtained by Wysin,\cite{19} $\lambda_c \approx 0.72$ for a square lattice, while for a hexagonal and a triangular lattice the values of the critical parameter are $\lambda_c \approx 0.86$ and $0.62$ respectively. Naturally, the numerical analysis was based on the discrete model in which the condition $M=0$ at the center of the vortex is not necessary since the center of the vortex in a numerical simulation is located at a point that does not coincide with the lattice site. We can assume that the length of the spin is the same for all lattice sites. Naturally, the problem of singularity does not emerge in this case. However, for a macroscopic description of intraplanar vortices based on differential and not difference equations, we must use vortex solutions with $|\mathbf{M}|=0$ at the center of the vortex. The contraction of the magnetization length does not contradict in principle the phenomenological theory of magnetism. This contraction can be associated with thermal or quantum-mechanical fluctuations and is described with the help of a phenomenological expression for the energy of a magnet taking into account the exchange energy.
The analysis does not depend on the physical reason behind spin contraction (quantum-mechanical or thermal fluctuations) and is determined only by the form of the function \( f(|M|) \). Bulaevskii and Ginzburg analyzed the domain walls at which magnetization changes over long distances and proved that such walls are advantageous from the energy point of view near the Curie point. Domain walls of this type in magnets with the quantum-mechanical contraction of spin were considered in Refs. 22, 23. The solution obtained in Refs. 24, 25 for a pointlike singularity ("hedgehog") of the magnetization field with \( |M| = 0 \) at the center describes a Bloch point. A static vortex of the magnetization field can be investigated similarly. An analysis of the static properties in the case of thermal and quantum-mechanical contractions of \( M \) is carried out in the same way. We will prove, however, that the dynamic properties of vortices differ significantly in these cases.

The Landau–Lifshitz equations are used for describing the dynamics of magnetization in Heisenberg ferromagnets for which quantum-mechanical fluctuations of spin are not significant at finite temperatures. However, these equations as a rule cannot describe time variations of the magnitude of the magnetization vector. Indeed, these equations have a local integral of motion \( \partial M^2 / \partial t = 0 \) even if we take into account relaxation terms in the Landau–Lifshitz or Hilbert form. A static solution with \( M^2 = M^2(r) \neq \text{const}, M(0) = 0 \) describing a stationary vortex does not contradict the Landau–Lifshitz equation. However, the motion of an intraplanar vortex cannot be described on the basis of this equation in view of the relation \( \partial M^2(r)/\partial t = 0 \). It should be noted that the situation with the motion of an intraplanar vortex changes insignificantly even in the generalized version of the Landau–Lifshitz equation proposed by Bar'yakhtar, in which the local integral of motion \( M^2 \) is destroyed. Indeed, according to the equation proposed in Ref. 26, the value of \( \partial M^2(r)/\partial t \) differs from zero, but its variation is of purely relaxation nature. The value of \( \partial M^2(r)/\partial t \) is small due to the smallness of the exchange relaxation constant \( \lambda_x \). As \( \lambda_x \propto T^2 \), the vertex velocity must tend to zero upon a decrease in \( T \), but in the numerical experiments the rms velocity of the ensemble of intraplanar vortices was even higher than that for extraplanar vortices. Thus, the vortex dynamics in the description based on the Landau–Lifshitz equations must be purely diffusive (overdamped). A detailed discussion of this problem is beyond the scope of this paper.

Let us consider the situation for magnets with strong effects of quantum-mechanical spin contraction. The validity of the Landau–Lifshitz equation has not been proved for strongly anisotropic magnets. Moreover, some results are in contradiction with this equation. By way of an example, we will consider a ferromagnet (FM) with spin \( s = 1 \) and with a strong one-ion anisotropy close to easy-plane anisotropy, which is known as a planar ferromagnet (PFM).

The static and dynamic properties of PFM were studied by many authors (see also the review article by Loktev and Ostrovskii) on the basis of the Heisenberg Hamiltonian with one-ion anisotropy:

\[
\hat{H} = - \sum_{\langle 1,1' \rangle} J(1-1') S_i S_{1'} + B \sum_i (S_i \tau_z)^2 - g \mu_0 \sum_i (S_i \mathbf{H})
\]

where \( J \) is the exchange integral, \( S_i \) the spin operator at the site, \( B \) the one-ion anisotropy constant, \( \mathbf{H} \) the magnetic field, and \( \langle 1,1' \rangle \) denotes the summation over nearest neighbors. In such a spin model, interesting effects cause quantum-mechanical fluctuations and not observed Heisenberg ferromagnets were predicted. For example, display nonmagnetic (quadrupole) phases with zero value of spin (but with a nonzero mean value of the number of spin projections), additional magnetic branches, and solitons (domain walls and vortices) of varying magnitude of magnetization. In particular, a model solution of the type of intraplanar vortices exists for PFM. A preliminary analysis of the dynamic properties of such vortices revealed that they significantly affect the properties of planar vortices in weakly anisotropic ferromagnets: their dynamics is not of the above-mentioned diffusive type as in the case of traditional weakly anisotropic ferromagnets.

Thus, the existence of intraplanar vortices is predicted by two models of similar in the sense of Heisenberg ferromagnets, but the contractive magnetization length is mainly due to thermal fluctuations. The magnetization dynamics in this model can be described on the basis of a modified Landau–Lifshitz equation with an "exchange" relaxation term in the form presented in Ref. 26, and the motion of vortices is of relaxation nature.

A model determining the properties of quanm mechanical PFM, the magnetization dynamics cannot be described on the basis of the Landau–Lifshitz equations, requires a special analysis. It should be emphasized that the condition of applicability of a macroscopic description (long-wave approximation) formulated in the form of an equality \( \Delta > \alpha \), where \( \Delta \) is the size of the vortex core and \( \alpha \) is the atomic spacing, can be satisfied for both models. Both ferromagnets, this condition can be written in the form of \( \langle M \rangle = M_0 \), where \( \langle M \rangle \) is the mean value of the magnetization and \( M_0 \) its nominal value. For a "classical" PFM, such a condition is satisfied near the Curie point, while for a quantum-mechanical PMF this condition is satisfied in the vicinity of a transition to the quadrupole phase (see below).

In this work, we analyze the structure of intraplanar vortices and their dynamic properties for quantum-mechanical PFM in the vicinity of the transition from the ordered to quadrupole phase, where the role of quantum-mechanical spin contraction is most significant. We will analyze Berezinskii–Kosterlitz–Thouless (BKT) transition for s magset.
For the model of a purely uniaxial PFM in the vicinity of the phase transition from the magnetic phase to the quadrupole phase, this Lagrangian was obtained in Refs. 30,33 from the microscopic Hamiltonian (2) using various methods. The magnetic field was taken into account in Ref. 31. Using the parameterization $s_x = s \cos \varphi, s_y = s \sin \varphi$, we can write the Lagrangian in the form

$$L = \frac{IR_0^2}{2a^2} \int d^2x \left\{ \frac{1}{c^2} \left( \frac{\partial s}{\partial t} \right)^2 - (\nabla s)^2 - \frac{(s^2 - s_0^2)^2}{R_0^2} \right\} + s_z \left\{ \frac{1}{c^2} \left( \frac{\partial \varphi}{\partial t} - gH \right)^2 - (\nabla \varphi)^2 \right\}. \tag{3}$$

Here $I = \Sigma J(1)$ is the mean value of the exchange integral, $R_0$ the characteristic scale of length, whose value is determined by the formula $R_0^2 = \Sigma J(1)^2/2J; s_0 = (1 - B^2/16)^{1/2} \approx 1$ is the equilibrium value of the average spin for $H = 0$; $c = (R_0/4\hbar)(B1)^{1/2}$ the characteristic velocity coinciding with the phase velocity of spin waves in the linear theory, $B \approx 4I$ and $c = 1R_0/2\hbar$ in the vicinity of the transition, and $H$ is the magnetic field perpendicular to the easy plane (for other directions of the field, the degeneracy in the orientation of the magnetization in the easy plane is removed, and a solution of the vortex type does not exist).

Using Lagrangian (3), we can obtain the following expressions for the integrals of motion (energy and momentum) of the magnetization field of the PFM:

$$E = \frac{IR_0^2}{2a^2} \int d^2x \left\{ \frac{1}{c^2} \left( \frac{\partial s}{\partial t} \right)^2 + (\nabla s)^2 + \frac{(s^2 - s_0^2)^2}{R_0^2} \right\} + s_z \left\{ \frac{1}{c^2} \left( \frac{\partial \varphi}{\partial t} \right)^2 + (\nabla \varphi)^2 \right\}, \tag{4}$$

where

$$P = \int d^2x (p_{LI} + p_\varphi), \tag{5}$$

where

$$p_{LI} = -\frac{IR_0^2}{a^2 c^2} \left( \nabla s \frac{\partial s}{\partial t} + s^2 \nabla \varphi \frac{\partial \varphi}{\partial t} \right),$$

$$p_\varphi = \frac{IR_0}{a^2 c} gH s^2 \nabla \varphi.$$  

The term containing $p_{LI}$ corresponds to the conventional Lorentz-invariant (LI) term, while the gyroscopic term $p_\varphi$ is due to the presence in the Lagrangian of a term linear in $\partial \varphi/\partial t$ and in magnetic field.

It will be shown below that the inclusion of viscous friction of a vortex is important in the description of the dynamics of intraplanar vortices. Dissipative processes can be taken into account by writing the dissipative function $Q(m, \dot{m})$ and by replacing the Lagrange equations $\delta L/\delta \dot{m} = 0$ and $\delta L/\delta \dot{m} = 0$ by the equations $\delta L/\delta \dot{m} - \delta Q/\delta \dot{m} = 0$ and $\delta L/\delta \dot{m} - \delta Q/\delta \dot{m} = 0$. However, while writing the dissipative function, we must take into account the dynamic symmetry of the system (these ideas for Heisenberg magnets were developed by Bar'yakhtar).25,34-38

The PFM model described by Hamiltonian (2) contains an exact integral of motion, viz., the total value of the projection $M_x \propto \dot{s}_x$ of the magnetic moment on the preferred axis. This integral of motion must naturally be preserved in the phenomenological description also. Its magnitude is proportional to $\dot{J}_z$, i.e., the total value of the $z$-projection of angular momentum whose density is defined as $\dot{J}_z = \partial L/\partial \varphi$. Consequently, the equation of motion for the quantity $\dot{J}_z \propto s^2(\dot{\varphi} - gH)$ must describe a conservation law in the divergent form even when dissipation is taken into account. This means that the dissipative function cannot contain terms of the type $f(s^2) \varphi^2$. In the simplest form, the corresponding term in $Q$ defining the relaxation term in the equation for $\partial J_z/\partial t$ can be chosen in the form $s^{2n}(\nabla \varphi)^2$, where $n$ is a positive integer.1) It should be noted that the absence of terms of the type $\varphi^2$ in the dissipative function of PFM was observed long ago by Halperin and Hohenberg,30 but the term $(\nabla \varphi)^2$ obtained by these authors is inapplicable for describing magnetic excitations with a significant variation of $s$ over the length (e.g., vortices) and is in contradiction to the results of calculations of magnon damping in PFM with a strong quantum-mechanical spin contraction.4) The dynamic equation for $s$ has no limitations of this type, and the corresponding term in the dissipative function can be presented in the simplest form $(\varphi/\partial t)^2$. Thus, we will proceed from the following form of the dissipative function:

$$Q = \frac{IR_0}{2a^2c} \int d^2x \left\{ \lambda_1 \left( \frac{\partial s}{\partial t} \right)^2 + \lambda_2 R_0 s^{2n} \left( \nabla \varphi \frac{\partial \varphi}{\partial t} \right)^2 \right\}, \tag{6}$$

where $\lambda_1$ and $\lambda_2$ are dimensionless relaxation constants. If we take the dissipative terms into consideration, the integrals of motion (4) and (5) break down (for example, $dE/\partial t = -2Q$).

**THE STRUCTURE OF VORTEX SOLUTIONS**

The equations of motion for $s$ and $\varphi$ which follow from Eqs. (3) and (6) can be presented in the form

$$\nabla^2 s - \frac{1}{c^2 \partial t^2} s \left( \nabla \varphi \right)^2 - \frac{1}{c^2 \partial t} \left( \frac{\partial \varphi}{\partial t} \right)^2 - 2gH \frac{\partial \varphi}{\partial t} = 0,$$

$$-\frac{2}{R_0} s (s^2 - s_0^2) = \frac{\lambda_1}{R_0 c} \frac{\partial s}{\partial t},$$

$$\nabla (s^2 \nabla \varphi) - \frac{1}{c^2 \partial t} \left( s^2 \frac{\partial \varphi}{\partial t} - gH \right) = -\frac{\lambda_2 R_0}{c} \nabla \left( s^{2n} \frac{\partial \varphi}{\partial t} \right). \tag{8}$$

We will use these equations for analyzing the structure and dynamics of magnetic vortices. The simplest case corresponds to a stationary vortex. It is described by a solution of the form

$$\varphi = \varphi_0 + \nu X, \quad s = s_0 f(\xi),$$

$$\xi = r/\Delta_0, \quad \Delta_0 = R_0 = R_0/s_0 \nu^2,$$

where $\nu = \pm 1, \pm 2, \ldots$ is the topological charge of the vortex, $r$ and $X$ are the polars coordinates in the plane of the magnet, and $f(\xi)$ is the Gross–Pitaevskii function which is determined by the solution of the ordinary differential equation

$$\frac{df}{d\xi} + \frac{1}{\xi} \frac{df}{d\xi} = \frac{\nu^2}{\xi^2} + (\nu^2 - 1), \tag{10}$$

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similar to the equation describing the vortex structure in a nonideal Bose gas\textsuperscript{42} (see also Ref. 16). The asymptotic behavior of this function is determined by the following relations: \( f(\xi) \propto \xi^{|\nu|} \) for \( \xi \to 0 \) and \( f(\xi) \sim 1 - v^2/2\xi^2 \) for \( \xi \to \infty \).

In the nondissipative limit, we can write exact solutions for a moving vortex. For a vortex moving at a constant velocity \( v = v_0 \), the solution for \( H = 0 \) can be obtained from the solution for a stationary vortex through the Lorentz transformation \( x' = x - vt(1 - v^2/c^2)^{-1/2}, y' = y \):

\[
\varphi = \varphi_0 + \nu \arctan \left( \frac{y}{x'} \right), \quad s = s_0 f(\xi'),
\]

where \( \xi' = \xi R/r, \) and \( R = \sqrt{x'^2 + y'^2} \).

The energy and momentum of such a vortex are determined by the LI formulas \( E_L(v) = E_0(1 - v^2/c^2)^{-1/2}, \) \( P_L(v) = (v/c^2)E_L(v) \), where \( E_0 \) is the energy of a static vortex. For a vortex with \( |\nu| = 1 \), we have

\[
E_0 = \frac{\pi I}{2} (R_0s_0/a)^2 \ln(0.41S/\Delta_0^2),
\]

where \( S \) is the PFM area per vortex.\textsuperscript{31} Thus, the static energy as well as the "effective mass" \( m_e = E_0/c^2 \) of the vortex for \( H = 0 \) diverge as \( \ln S \).

In the case when \( H \neq 0 \), the Lorentz invariance is violated. However, in this case also we can construct a solution describing a moving vortex.\textsuperscript{31} The form of the function \( s(\xi) \) does not change in this case, and the expression for the function \( \varphi \) acquires an additional term proportional to \( kr - \omega t \), which is typical of the vortex dynamics for superfluid systems.\textsuperscript{43} Heisenberg ferromagnets,\textsuperscript{17} and antiferromagnets\textsuperscript{44}.

\[
\varphi = \varphi_0 + \nu \arctan \left( \frac{y}{x'} \right) + k \cdot r - \omega t, \quad s = s_0 f(\xi').
\]

\[
k = \frac{gH}{c^2 \sqrt{1 - v^2/c^2}}, \quad \omega = gH \left( \frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right).
\]

For low vortex velocities \( v \ll c \), the value of \( \omega \propto v^2 \), and the main effect of the field is the emergence of a finite value of \( \nabla \varphi \) as \( r \to \infty \).

Using Eq. (13), we can easily describe the extraplanar component of the spin:

\[
s_z = - \hbar s z \left( \frac{\partial \varphi}{\partial t} - \frac{gH}{c} \right) = \hbar s_0 f(\xi') \frac{gH - v
u y R}{c^2 \sqrt{1 - v^2/c^2}}.
\]

It can be easily seen that the quantity \( s_z \) contains two terms. One term is due to the magnetic field and is a sign-definite function of coordinates. The other term determined by dynamic effects is proportional to the vortex velocity; its contribution has different signs for \( y > 0 \) and for \( y < 0 \).

The vortex energy for \( H \neq 0 \) contains an additional term proportional to \( v^2 H^2 S \). In the nonrelativistic limit of energies, \( E(\nu) = E_L(v) + \mu \nu)^2/2 \), where the quantity \( \mu = (I/R_0s_0/a)^2 H^2 S/4\hbar \) is proportional to the area of the vortex can be regarded as the vortex mass. It should be noted, however, that this quantity diverges as \( S \) rather than \( \ln S \) and essentially determines the mass of the cond moving together with the vortex.

We have obtained the exact solution describing the motion of a solitary vortex in the presence of a nonlinear condensate \( \nabla \varphi = k = \text{const} \cdot v \) in a nondissipative dium. A typical feature of this solution for \( H \neq 0 \) is a single-valued relation between the vortex velocity \( v \) at a given value of the angular variable \( \varphi \) away from the vortex; \( v \ll c \), this relation is described by the formula

\[
\nabla \varphi = \nu gH/c^2.
\]

Such a single-valued relation is typical of many of ordered media like superfluid systems\textsuperscript{43} or Heisenberg ferromagnets with extraplanar vortices.\textsuperscript{17} A peculiar feature of the given model (as well as a Heisenberg antiferromagnet) is that the coefficient in this formula is proportional to the magnetic field \( H \) and vanishes in the limit of the LI \( H = 0 \).

Relation (15), i.e., the single-valued relation between \( \nabla \varphi \) and \( v \), is also valid when dissipative processes are taken into account, and when an exact solution cannot be obtained. In order to derive this equation, we multiply Eq. (7) \( b \) and Eq. (8) by \( \nabla \varphi \), sum the obtained relations, and integrate the result over the region of space occupied by the vortex. After simple transformations, we obtain

\[
\frac{1}{c^2} \left( - \frac{\partial^2 s}{\partial t^2} \nabla \varphi - \frac{\partial s}{\partial t} \nabla \varphi - s^2 \frac{\partial^2 \varphi}{\partial t^2} \nabla \varphi \right) + \frac{\partial^2 \varphi}{\partial t^2} s \nabla \varphi \left( \frac{gH/c^2}{\nabla \varphi} \right) \left( \frac{\partial s^2}{\partial t} \nabla \varphi \right) - \frac{\partial s}{\partial t} \nabla \varphi \left( \frac{gH c^2}{\nabla \varphi} \right) = \frac{1}{R_0c} \left( \lambda_1 \frac{\partial s}{\partial t} \nabla \varphi - \lambda_2 R_0^2 \nabla \varphi \right) \varphi(R_0^2 \nabla \varphi - \frac{\varphi}{\nabla \varphi} \right) \nabla \varphi \left( \frac{gH/c^2}{\nabla \varphi} \right) \left( \frac{\partial s^2}{\partial t} \nabla \varphi \right) \nabla \varphi \left( \frac{gH c^2}{\nabla \varphi} \right) \left( \frac{\partial s}{\partial t} \nabla \varphi \right)\right)

where we have introduced the notation \( \langle F \rangle = \int F d^2x \) (density). The above solutions suggest that for \( v \ll c \) and for \( s \) but nonzero values of \( \lambda_1 \) and \( \lambda_2 \), the solution can be written to a high degree of accuracy by the formula

\[
\varphi = \varphi_0 + \nu \arctan \left( \frac{y - Y(t)}{x - X(t)} \right) + k \cdot r,
\]

where the function \( f(\xi) \) is defined in Eq. (X) \( X(t) = X(t) + Y(t) \), and is the coordinate of the vortex center, and \( k \) is the value of the gradient of \( \varphi \) away from the vortex. Substituting these test functions into Eq. (16) and evaluating the integrals, we can obtain a relation between the vortex velocity \( \partial X/\partial t \), its acceleration, and the quantity \( k \). The term appearing in formula (16) has a clear physical meaning.
and the expression on the whole can be regarded as the balance equation for forces acting on the vortex.

The first term contains the summands proportional to the vortex acceleration as well as the summands nonlinear in the velocity. The latter summands will be disregarded. As regards the inertial term, it can be reduced to the form

\[ m_s \frac{\partial^2 \mathbf{X}}{\partial t^2}, \]

where \( m_s = E_0 / c^2 \), is described by the Lorentz-invariant formula (12), and contains a logarithmic divergence. This is in accord with recent calculations made by Wyssin et al. and Vökel et al., but does not agree with the results obtained by Mertens et al., who realized numerical simulation of the rotational dynamics of a pair of vortices in the circular geometry and obtained the value of effective mass for an extraplanar vortex in a Heisenberg ferromagnet at \( T = 0 \) from an analysis of the data on frequency and radius of motion. This value was characterized by a linear dependence on the size of the system. An analytic description of inertial effects in the dynamics of the vortex ensemble in PFM has not yet been constructed. The numerical simulation carried out in Refs. 9–15 does not reveal any influence of inertial effects on the dynamics of the vortex ensemble for not very low temperatures (of the order of \( T_c \)). For this reason, we will not discuss inertial effects here.

The remaining terms determine the effective equation of motion of a vortex in the chosen approximation (i.e., disregarding the inertia of vortices and terms of the type \( (\partial \mathbf{X}/\partial t)^2, (\partial \mathbf{X}/\partial t)^3 \), etc.). It can be easily verified that the term containing \( k \) makes no contribution to the second addend which can be reduced to the form

\[ G \left[ \frac{\partial \mathbf{X}}{\partial t} \times \mathbf{e}_1 \right], \]

where

\[ G = -2 \pi \nu R_{0}^{2} \rho^{2} H / c^{2} a^{2} = -16 \pi \hbar \mu_0 \nu_0 H / a^2. \]

The term containing \( G \) has the meaning of the gyroscopic force acting on the vortex. Such terms are well known for Heisenberg FM. In the case of a PFM, the gyroscopic constant \( G \) exists only in the presence of a magnetic field. The smallness of the constant \( G \) and its vanishing for \( H \to 0 \) determines significant peculiarities in the dynamics of a vortex ensemble in PFM as compared to a Heisenberg FM (see below). It should be noted that the dependence \( G \propto H \) is typical of vortices in Heisenberg antiferromagnets whose dynamics were studied in Ref. 44.

The integral appearing in the third addend in (16) can be reduced to the integral over the contour of the region containing a vortex. For a low-density gas of vortices, the size \( L \) of this region can be regarded as large. In the limit \( L \to \infty \), only the contribution proportional to \( k \) remains finite and can be presented in the form

\[ F_k = 2 \pi \nu \left( \frac{R_{0}^{2}}{a} \right)^2 [k \times \mathbf{e}_1]. \]

The physical meaning of this term is quite obvious when the gradient of the phase at a point of the given vortex is due to the presence of the remaining vortices constituting the ensemble. In this case, we obtain the following expression for \( k \) at the point \( r \): \( k(r) = \sum_j \nabla \varphi^{(j)}(r - R_j) \), where \( X \) are the coordinates of vortices and \( \varphi^{(j)}(r) \) corresponds to stationary vortex solutions. In this case, the force \( F_k \) defined as

\[ F_k = -\nabla H_{\text{int}} = e\mathbf{E}, \]

\[ H_{\text{int}} = -2 \sum_{i \neq j} e_i e_j \ln |X_i - X_j|, \]

has the form of the force \( v_i(R_{0} R_{0} / a) \sqrt{\pi} \) of the 2D Coulomb interaction between the charges located at the points \( X_i \), and the quantity \( e \) is similar to the electric field acting on the given charge (vortex) and produced by other vortices.

It should be noted that if \( \nabla \varphi \) in formula (15) defining the relation between the vortex velocity and the gradient of phase for a vortex ensemble at infinity is replaced by its specific value given above, we obtain the following relation:

\[ gH = \frac{X_j}{c^2} \frac{\partial}{\partial t} = \sum_j \nu l \left[ e_i \times \frac{X_i - X_j}{|X_i - X_j|^2} \right]. \]

Finally, the calculation of the last term in formula (16) leads to the following expression coinciding with the standard expression for the force of viscous friction:

\[ F_{\text{vis}} = -\eta \frac{\partial \mathbf{X}}{\partial t}, \quad \eta = \frac{2Q}{\nu^2}, \]

where the viscosity \( \eta \) is defined as the sum of two terms:

\[ \eta = \frac{\pi l R_{0}^{2}}{c a^2} (\lambda_1 \gamma_1 + \gamma_2 \gamma_2), \]

where

\[ \gamma_1 = \frac{1}{2} \int \xi d\xi J^{2} f^{2n+1} \]

\[ \gamma_2 = 4 \nu \int d\xi f^{2n-1} \frac{\xi^2}{\xi^2}. \]

Note that for \( n = 1 \) the second term contains a logarithmic divergence. For a gas of vortices with a finite density \( n_0 \), this logarithmic factor is replaced by a finite quantity \( \ln(\xi / \Delta_0) \approx \ln(n_0 / \Delta_0) \), where \( \xi \) is the average distance between vortices and \( \Delta_0 \) the radius of the vortex core.

Thus, we arrive at the following effective equation of motion for an ensemble of vortices:

\[ \frac{\partial X_i}{\partial t} + \sum_j e_i e_j \frac{X_i - X_j}{|X_i - X_j|^2} \frac{\partial X_i}{\partial t} = 0. \]

It can be easily seen that Eq. (24) has the same meaning (i.e., determines the relation between \( \mathbf{v} \) and \( \nabla \varphi \)) as formula (20) indicated above for the case \( \lambda_1, \lambda_2 = 0 \), but contains a definite value of \( \nabla \varphi \) and takes relaxation into account. Equations of motion of the type (24) for Heisenberg FM were obtained by various methods. The most rigorous derivation is presented by Nikiforov and Sonin who used the soliton theory of perturbations (similar to that developed in Ref. 47). These methods can also be applied for PFM. Under the same assumptions as in Refs. 3, 17, 27, they lead to Eq. (24).
VOXEL THERMODYNAMICS OF A PLANAR FERROMAGNET

Thermodynamic properties of 2D PFM in the vicinity of the BKT transition are determined by the presence of intraplanar vortices. Berezinskii\textsuperscript{1} and Kosterlitz and Thouless\textsuperscript{2} found that below $T_c$ (in the Berezinskii phase), soliton pairing accompanied by the formation of coupled vortex–antivortex pairs takes place, while at $T>T_c$ (in the topological phase) vortex pairs dissociate. The presence of free vortices is manifested in spin correlation over a finite length. The calculations made for the XY-model based on the renormalization group give

$$\xi(T) = \xi_0 \exp(b \tau^{-1/2}), \quad \tau = (T - T_c)/T_c,$$  \hspace{1cm} (25)

where $b = \text{const}$ and $\xi_0$ is the quantity of the order of the magnetic length $\Delta_0$ (see Refs. 2,9). The correlation length $\xi(T)$ plays the role of the average distance between vortices and ensures their finite density $n_v$ which is defined by the formula $n_v \approx (2\xi)^{-2}$.\textsuperscript{3,4,45}$\textsuperscript{46}$ Following Ref. 2, we estimate the transition temperature $T_c$. The change in the free energy of the system associated with the creation of a vortex is $\Delta F = E_0 - TS_0 \approx 0$, where $E_0$ is the vortex energy (12) and $S_0 = \ln S$ is the entropy of indeterminacy in the position of the vortex. Taking into account the fact that the vortex energy is also proportional to $\ln S$, we obtain $\Delta F = (T_c - T) \ln S$; consequently, the value of $\Delta F < 0$ for $T > T_c$, and the creation of a vortex becomes advantageous from the energy point of view. Thus, we can obtain the following estimate for the temperature of the BKT-transition: $T_c \approx 1/2 \pi l(R_0 \delta_0/a)^2$. As the mean value of spin $s_0$ decreases due to quantum fluctuations ($s_0 \ll 1$ in our case), the transition point $T_c$ for a quantum ferromagnet with spin $s = 1$ is displaced towards lower temperatures. In the case of a PFM in the vicinity of the transition from the ordered to the quadrupole phase, the value of $s_0 \ll 1$, the value of $T_c$ can be much smaller than the value $T_c \approx 1$ which is standard for Heisenberg FM, and the transition occurs due to quantum–mechanical contraction of spin at very low temperatures. Besides, the value of $s_0$ varies significantly under the action of a magnetic field. We can easily estimate this dependence proceeding from Eq. (3): $s_0(H) = -B^2/16l^2 + 1/2g\chi(H_0/c)^2$, which means that the value of $T_c$ strongly depends on the magnetic field:

$$T_c = \frac{\pi l}{2} \left( \frac{R_0}{a} \right)^2 \left[ 1 - \frac{B^2}{16l^2} + \frac{(gH_0)^2}{2c^2} \right].$$  \hspace{1cm} (26)

Let us calculate the contribution of vortices to the PFM response functions. A similar theoretical analysis was carried out in a number of publications for Heisenberg magnets.\textsuperscript{3,9,15}$\textsuperscript{15}$

For example, the effective equation of motion for vortices of the type (24) was used by Huber\textsuperscript{3} for a thermodynamic analysis of the velocity of a vortex gas in a Heisenberg FM. The calculations made by Mertens et al.\textsuperscript{9} for the contribution of the vortex ensemble to the response functions of a Heisenberg FM revealed that the vortices make a significant contribution to the CP region. The main parameters determining the form and the width of the central peak are the density of vortices and the rms velocity $\bar{u}$ of the vortex gas. In the case of PFM, the thermodynamic characteristics of the gas of vortices acquire a singularity associated with the possibility $G$ vanishes as $H \rightarrow 0$.

Let us now calculate $\bar{u}$. Using Eq. (24), we can obtain the linear relation between the vortex velocity and the effective “electric field” $E$ introduced above (19) and describing the effect of the remaining $\nu (\nu^2 + G^2) = c(-\nu E + G(e_\chi X E))$. Therefore, the value of the square of velocity for the given vortex expressed in terms of the correlators of the field $E$, correlators were calculated in Ref. 49 for a self-consistent field of a 2D electron plasma in the approximatum leading centers of orbits. Their form does not depend constants $G$ and $\eta$ and is determined by the formula $\langle E_x E_y \rangle = \langle E_z \rangle^2 \delta_{x y} = n_v \pi e^2 / 4 \ln \Lambda$, where $\alpha, \beta = \eta$, $\alpha$ is the equilibrium density of vortices, and $\Lambda$ charac the closeness of the gas of vortices to thermodynamic librum. For the limiting cases of random and equilibria distributions, the quantity $\Lambda$ is defined respectively $\Lambda_R = \pi a^2$ and $\Lambda_p = 4\pi R^2 n_v e^2 / a^2$, $S$ being the area system, and $T$ the temperature.

Using the relation between $\bar{u}$ and $E$ as well as the expression for $\langle E_x \rangle$, we can easily find the rms veloc vortices:

$$\bar{u} = \frac{c \langle E_x \rangle^{1/2}}{(G^2 + \eta^2)^{1/2}} = \frac{c}{c} \left( \frac{n_v R_0 H_x}{H^2 + H_0^2} \right)^{1/2} \ln \Lambda,$$

where the characteristic (exchange) field $H_x = 1/2 \mu_0 a$ and the relaxation field $H_0 = \lambda_1 \gamma H_x$ have been introduced for venuence.

Thus, for $H \gg H_0$, the velocity $\bar{u} \approx H_0/2H$ is inver proportional to the magnetic field $H$, while for $H \ll H_0$, it does not depend on the field strength: $\bar{u} \approx H_0/2H$. Unfortunately, a micr analysis for dissipative constants in PFM has not been ried out yet, and hence we cannot estimate the $H_x / H_0$. It should only be noted that for $H < H_0$, (in par lar, for $H = 0$), the value of $\bar{u}$ is inversely proportional to a relaxation constant. As $\eta$ usually increases with $T$, this mula predicts a decrease in $\bar{u}$ with increasing $T$ (rather its increase as in the Heisenberg ferromagnet considere Huber\textsuperscript{4} in which $\bar{u} \propto (E_x)^{1/2}/(G)$).

Let us now calculate the vortex contribution to the namic structural factor. The extrapolator DSF components locally sensitive to the presence of vortices in view of localization of the component $s_\tau(r)$ in the vicinity of vortex center. The central peak for extrapolator correlati has the Gaussian shape. The peak intensity increases with the number of vortices and depends significantly on the vort form factor. The intraplanar DSF components are glob sensitive to the presence of vortexes destroying the lo range order in the plane of the 2D magnet. The Lorentz CP for intraplanar correlations does not depend on the vor form factor. The CP intensity decreases upon an increase the number of vortices because the magnet becomes homegeneous on the average.

The calculation of the vortex contribution to the DSF made in analogy with the analysis carried out by Mert et al.\textsuperscript{9} (see also Refs. 13–15), and we will not consider he

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the details of this calculation. The central peak widths are
\[ \Delta \Gamma = q^2 \tilde{u} \quad \text{and} \quad \Delta \Gamma_z \approx 1.14 \tilde{u} n_v^{1/2} \]
both for extra- and intraplanar components. 

The shape of intraplanar components does not depend on the field and is determined by expressions similar to those obtained in Refs. 9, 13, 15; for example, the formula
\[ S^{(0)}(q, \omega) = \frac{\kappa^2}{8 \pi n_v [\omega^2 + \kappa^2 (1 + q^2/4n_v)]^2} \]
determines the well-known expression \( \kappa = \tilde{u}(\pi n_v)^{1/2} \) for the Lorentz-type CP.

The extraplanar DSF components form the Fourier transform of the correlator  \( \langle \sigma_x(r,t) - \sigma_x(0,0) \rangle \), where \( \sigma_x^{(0)} = \hbar \sigma_0 H \).

Carrying out the space–time Fourier transformation, we arrive at the following expression for extraplanar DSF components:
\[ S^{(1)}(q, \omega) = \frac{\hbar^2 n_v}{2 \pi \Delta_0^{1/2} q^2} \left( \frac{2H^2}{F_1(q)} \right)^2 \]
\[ + \frac{\tilde{u}^2}{2 \Delta_0} \left( \frac{F_2(q)}{2} \right)^2 \exp\left(-\frac{i\omega q^2}{2} \right). \]

The first term, which is determined by the static contribution to magnetization, and the second term, which is associated with the dynamic contribution (see Eq. (14))
\[ F_1(q) = 2\pi \int_0^\infty \frac{dr}{\Delta_0} J_0(qr) \left| \frac{s^2 - s^2(r)}{s^2} \right|, \]
\[ F_2(q) = 2\pi \Delta_0 \int_0^\infty \frac{dr}{\Delta_0} J_1(qr) s^2(r), \]
are the two form factors of the vortex (static and dynamic respectively), and \( J_1(x) \) is Bessel’s function of the 1st order. Approximating the Gross–Pitaevskii function by the expression \( f(x) \approx x(1 + x^2)^{-1/2} \), we obtain the following estimates for the form factors (30):
\[ F_1(q) = 2\pi R^2 \kappa q_r q_0 / s_0 \tilde{v}_2, \]
\[ F_2(q) = 2\pi R^2 \kappa q_r q_0 / s_0 \tilde{v}_2, \]
where \( \kappa_i(x) \) is the ith order Macdonald function. The analysis of the magnetic field dependence of the DSF extra-planar component can serve as a good test for comparing the theory and experiment.

CONCLUSION

The analysis carried out in this work reveals a significant difference of the vortex dynamics in PFM from those in a Heisenberg FM and a strong effect of the applied magnetic field perpendicular to the easy plane. This is manifested in the following peculiarities.

The vortex dynamics in PFM in zero transverse magnetic field is described by the LL equations, and the value of rms velocity of vortices is determined by viscosity and strongly depends on temperature.

In the presence of the field, the effects of “freezing” of vortices in the condensate and of their gyroscopic motion are intensified. With increasing field, a transition from viscous gyroscopic motion takes place, which is manifested in \( \tilde{u} \)

and is responsible for the field dependences of the characteristics of peaks. The field also makes a special contribution to extraplanar DSF components whose intensity considerably depends on \( H \).

The authors are grateful to V. G. Bar'yakhtar and Yu. N. Mitsai for a discussion of this work and to A. N. Kichzhiev for collaboration. This research was partly supported by grant No. 2.3/194 from the Ukrainian State Committee on Science and Technology and grant No. UB7000 awarded by the International Science Foundation.

The value of \( n \) can be determined, for example, from a macroscopic analysis of spin wave damping and from a comparison of this value with the result obtained from the given dissipative function. However, such an analysis has not yet been carried out for a purely easy-plane PFM. The results obtained in Ref. 41 for a PFM with a finite anisotropy \( \beta \) in the basal plane are not applicable for describing this term in the case where \( \beta = 0 \).

A similar divergence is also observed for the Hilbert relaxation term in a Heisenberg ferromagnet, but it is absent (see Ref. 50) in the relaxation terms proposed by Bar'yakhtar. We will not discuss the meaning of this divergence because the value of \( n \) for the given model is unknown.

REFERENCES

the details of this calculation. The central peak widths are \( \Delta \Gamma = \frac{\Delta}{4u} \) and \( \Delta \Gamma_x \approx 1.14u n_v^{1/2} \) both for extra- and intraplanar components.

The shape of intraplanar components does not depend on the field and is determined by expressions similar to those obtained in Refs. 9, 13, 15; for example, the formula

\[
S^{(2)}(q, \omega) = \frac{q N_{\text{eff}}}{8 \pi n_v \omega^2 \kappa^2 (1 + q^2 / 4n_v)} \left( 1 + \frac{q^2}{4n_v} \right)^2
\]

(28)
determines the well-known expression \( \kappa = \hat{u} (\pi n_v)^{1/2} \) for the Lorentz-type CP.

The extraplanar DSF components form the Fourier transform of the correlator \( \langle s_{(r,\tau)}(x) - s_{(0,0)}^{(0)} \rangle \), \( s_{(0)} = \hat{s}_{\parallel} H \). Carrying out the space-time Fourier transformation, we arrive at the following expression for extraplanar DSF components:

\[
S^{(2)}(q, \omega) = \frac{\hat{u}^2}{2 F_2(q)} \left[ e^{2H^2} + \frac{\omega^2}{2 \Delta_0} F_2(q)^2 \right] \exp(-\omega^2 / g^2 \hat{u}^2).
\]

(29)

The first term, which is determined by the static contribution to magnetization, and the second term, which is associated with the dynamic contribution (see Eq. (14))

\[F_1(q) = 2 \pi \int_0^\infty dr J_0(qr) [s^2_0 - s^2(r)],\]

\[F_2(q) = 2 \pi \Delta_0 \int_0^\infty dr J_1(qr) s^2(r),\]

are the two form factors of the vortex (static and dynamic respectively), and \( J_0(x) \) is Bessel’s function of the 0th order. Approximating the Gross–Pitaevskii function by the expression \( f(x) \approx x(1 + x^2)^{-1/2} \), we obtain the following estimates for the form factors (30):

\[F_1(q) = 2 \pi R^2_0 K(q R_0 / s_0 V_2),\]

\[F_2(q) = 2 \pi \Delta_0 K(q R_0 / s_0 V_2),\]

where \( K_i(x) \) is the \( i \)th order Macdonald function. The analysis of the magnetic field dependence of the DSF extra-planar component can serve as a good test for comparing the theory and experiment.

**CONCLUSION**

The analysis carried out in this work reveals a significant difference of the vortex dynamics in PFM from those in a Heisenberg FM and a strong effect of the applied magnetic field perpendicular to the easy plane. This is manifested in the following peculiarities:

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In the presence of the field, the effects of “freezing” of vortices in the condensate and of their gyroscope motion are manifested. With increasing field, a transition from viscous gyroscope motion takes place, which is manifested in \( \hat{u} \)

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