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Local Magnon Modes and the Dynamics of a Small-Radius Two-Dimensional Magnetic Soliton in an Easy-Axis Ferromagnet

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The internal dynamics of a small-radius precession magnetic soliton is considered. A variational formulation of the problem on the soliton–magnon interaction is proposed and used to calculate the frequency of a truly local mode. It is shown that this mode, as well as the conventional translational mode, remains localized in the small soliton radius limit. The presence of the local mode is confirmed by the numerical solution of the scattering problem. © *2005 Pleiades Publishing, Inc.*

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It is well known that topological solitons play an important role in the physics of two-dimensional (2D) magnetism [1]. Solitons have a profound impact on the thermodynamics and response functions of a magnet. The interest in 2D topological magnetic solitons had mainly been initiated by Belavin and Polyakov [2], who constructed a static solution with finite energy for a purely isotropic ferromagnet and showed that, in such a magnet, a thermal excitation of solitons destroys the long-range order. The inclusion of a magnetic anisotropy, which is always present in actual magnets, violates the scale invariance of the problem and gives rise to the characteristic magnetic length $l_0 = \sqrt{A/K}$, where A is the inhomogeneous exchange constant and K is the anisotropy constant. In an anisotropic ferromagnet, static non-one-dimensional solitons with finite energy are unstable with respect to collapse [3]. However, in easy-axis ferromagnets, the conservation of the total z projection of magnetization

$$N = \frac{S}{a^2}\int d^2x(1-\cos\theta)$$

leads to the presence of stable dynamic solitons with a precession of the magnetization vector with frequency Ω about the easy axis of the ferromagnet [4, 5]. In terms of angular variables for the normalized magnetization $\mathbf{m} = \mathbf{S}/S = (\sin\theta\cos\phi; \sin\theta\sin\phi; \cos\theta)$, the structure of such a precession topological soliton is described by the formulas [5]

$$\theta_0 = \theta_0(r), \quad \phi_0 = \varphi_0 + q\chi - \Omega t, \tag{1}$$

where r and χ are the polar coordinates in the magnet plane, φ_0 is an arbitrary angle, and the integer number q determines the π_2 topological charge of the soliton (below, we assume that $q = 1$). The structure of a stationary soliton is determined by the solution to the differential problem

$$\frac{d^2\theta_0}{dr^2} + \frac{1}{r}\frac{d\theta_0}{dr} - \sin\theta_0\cos\theta_0\left(\frac{1}{l_0^2} + \frac{1}{r^2}\right) + \frac{\Omega}{\omega_0}\sin\theta_0 = 0, \tag{2}$$
$$\theta_0(0) = \pi, \quad \theta_0(\infty) = 0.$$

Here, $\omega_0 = \gamma H_a$ is the gap in the spectrum of linear magnons characterized by the dispersion law $\omega(\mathbf{k}) = \omega_0(1 + l_0^2\mathbf{k}^2)$, γ is the gyromagnetic ratio, H_a is the anisotropy field, and l_0 is the magnetic length introduced above. In the isotropic case ($l_0 \longrightarrow \infty$), this equation determines the static ($\Omega = 0$) Belavin–Polyakov solution $\tan\theta_0/2 = R/r$, where the arbitrary parameter R has the meaning of the soliton radius. A solution to problem (2) can easily be constructed by the numerical method. Approximately, this solution is described by the simple function [6]

$$\tan\frac{\theta_0}{2} = \frac{R}{r}\exp\left(-\frac{r-R}{r_0}\right), \quad r_0 = l_0\sqrt{\frac{\omega_0}{\omega_0 - \Omega}}, \tag{3}$$

which actually determines the Belavin–Polyakov solution with the cutoff radius r_0. For an anisotropic ferromagnet, the energy of the soliton, its precession frequency Ω, and its characteristic radius are determined by the number $N \simeq S(R/a)^2$. Since the magnetic length is $l_0 \gg a$, where a is the lattice constant, the macroscopic approximation and the semiclassical description

$N \gg 1$ are also applicable to solitons with small radii $R \ll l_0$, which are discussed in this paper.

The properties of 2D precession solitons are well understood [1]. However, the laws governing the translational dynamics of solitons, i.e., the motion of a soliton as a whole, remain poorly investigated. The same problem exists for other 2D nonlinear states with nontrivial topology, such as, for example, magnetic vortices. The use of a combination of direct numerical modeling with analytical methods showed that the dynamics of vortices is non-Newtonian: the coordinate of the center of a vortex $\mathbf{X}(t)$ satisfies an equation that contains higher-than-second derivatives of $\mathbf{X}$ with respect to t [7] and coefficients depending on the size and shape of the ferromagnet sample in which the vortex moves. For localized solitons (1), the possibility of describing their dynamics on the basis of Newtonian equations with a finite effective mass is open to question. For example, in [8], it is stated that only the inertialess dynamics of a localized soliton "frozen into" the external spin flux is possible.

At first glance, the problem can easily be solved on the basis of the adiabatic perturbation theory for solitons [9]. This theory was used to describe the dynamics of one-dimensional solitons under the effect of arbitrary perturbations slowly varying in space and time. However, the applicability of this approach to 2D topological solitons is limited by the fact that, because of the nontrivial topology of magnetization distribution in the soliton, the dynamics of the soliton as an object possesses gyroscopic properties with the gyroscopic constant $G = 4\pi q\hbar S/a^2$ [1]. Therefore, one can expect that the simplest effective equation of motion written for the coordinate $\mathbf{X}$ of the soliton center and taking into account the inertial terms has the form

$$m_* \frac{d^2\mathbf{X}}{dt^2} = \mathbf{F}_g + \mathbf{F}_{\text{ext}}, \quad \mathbf{F}_g = G\left[\mathbf{e}_z \times \frac{d\mathbf{X}}{dt}\right], \tag{4}$$

where m_* is the effective mass of the soliton and $\mathbf{F}_{\text{ext}}$ is the external force acting on the soliton (e.g., from the boundary). Even for a free soliton ($\mathbf{F}_{\text{ext}} = 0$), this equation involves a "rapid" motion with the frequency $\omega_L = G/m_*$ (an analogue of the Larmor precession of a charged particle in magnetic field); hence, the condition of the slowness of the magnetization variation is not satisfied and the adiabatic perturbation theory [9] is inapplicable.

In this paper, we analyze the soliton dynamics by using a different approach: it was used in studying magnetic vortices and was based on considering the soliton–magnon scattering [10]. Its main idea is as follows: one considers the full set of magnon eigenmodes in the presence of a soliton, selects the eigenmodes that may be associated with the displacement of the soliton as a whole, and then compares the frequencies of these modes with the eigenmodes of Eq. (4). Thus, we verify the Newtonian equation and calculate the effective soliton mass m_*, which remains finite for any soliton radius but increases as the soliton radius R decreases.

To study the interaction of magnons with a soliton, we use the Landau–Lifshitz equations

$$\begin{gathered} l_0^2\nabla^2\theta - \sin\theta\cos\theta[1 + l_0^2(\nabla\phi)^2] = \frac{\sin\theta}{\omega_0}\frac{\partial\phi}{\partial t}, \\ l_0^2\nabla\cdot(\sin^2\theta\nabla\phi) = -\frac{\sin\theta}{\omega_0}\frac{\partial\theta}{\partial t}. \end{gathered} \tag{5}$$

For analyzing the soliton–magnon interaction, we consider small oscillations of magnetization (θ, ϕ) on the background of a stationary soliton (θ_0, ϕ_0). These oscillations can be described by the complex "wave function" $\psi = \theta - \theta_0 + i\sin\theta_0(\phi - \phi_0)$. Linearized equations for ψ have the form of the so-called generalized Schrödinger equation [6, 11]

$$\frac{i}{\omega_0 l_0^2}\partial_t\psi = H\psi + W\psi^*, \quad H = (-i\nabla - \mathbf{A})^2 + U. \tag{6}$$

The specific feature of this equation is the presence of the term involving W, which relates the solutions with positive and negative frequencies. One should also notice the appearance of the term that has the structure of the effective magnetic field with a vector potential $\mathbf{A}(\rho) = \mathbf{e}_\chi q\cos\theta_0/r$ proportional to the topological charge q. This term is caused by the gyroscopic properties of the medium and is related to the topological properties of the soliton.

The potentials involved in Eq. (6) have the form

$$U = \frac{\cos\theta_0}{l_0^2}\left[\cos\theta_0 - \frac{\Omega}{\omega_0}\right] - \frac{\sin\theta_0^2}{2}\left(\frac{1}{l_0^2} + \frac{q^2}{r^2}\right) - \frac{\theta_0'^2}{2},$$

$$W = \frac{\sin\theta_0^2}{2}\left(\frac{1}{l_0^2} + \frac{q^2}{r^2}\right) - \frac{\theta_0'^2}{2}.$$

To solve Eq. (6), we use the expansion in partial waves [10, 6, 11]:

$$\psi = \sum_{\omega, m}(u_m e^{i\Phi_m} + i v_m e^{i\Phi_m}), \quad \Phi_m = m\chi - \omega_m t + \eta_m. \tag{7}$$

This method allows us to reduce the generalized Schrödinger equation (6) to the spectral problem

$$\begin{gathered} \mathcal{H}|\mathbf{m}) = \omega_m|\mathbf{m}), \quad \mathcal{H} = \begin{Vmatrix} H_+ & -W \\ W & -H_- \end{Vmatrix}, \\ |\mathbf{m}) = \begin{Vmatrix} u_m \\ v_m \end{Vmatrix}. \end{gathered} \tag{8}$$

Here, $H_\pm = -\nabla_r^2 + U + (|\mathbf{A}| \pm m/r)^2$ are 2D radial Schrödinger operators that have no negative eigenvalues and the integer m has the meaning of the azimuthal quantum number. Note that spectral problem (8) for the matrix Hamiltonian $\mathcal{H}$ fundamentally differs from the standard set of coupled Schrödinger equations: the operator $\mathcal{H}$ is Hermitian only in the Hilbert space with indefinite metric [11]:

$$(\mathbf{m}|\mathbf{m}) = \int_0^\infty (u^2 - v^2) r dr.$$

In this case, for all of the eigenvalues ω_m, the operator H_- has a resolvent and, hence, $v_m = (H_- + \omega_m)^{-1} W u_m$ is a slave variable of the spectral problem.

In the absence of the soliton, the magnon amplitudes are $u_m = J_{|m|}(kr)$ and $v_m = 0$. The interaction with the soliton leads to the scattering of magnons. Then, away from the soliton, the magnon modes have the form

$$u_m \propto J_m(kr) + \sigma_m(k) Y_m(kr). \quad (9)$$

Here, $J_n(x)$ and $Y_n(x)$ are the Bessel and Neumann functions of order n, k is the wave number, $\sigma_m(k) = -\tan\delta_m(k)$ is the scattering amplitude, and $\delta_m(k)$ is the scattering phase. In the presence of the soliton, the formation of localized (bound) states is also possible. Distribution (7) includes all types of perturbations of the stationary soliton, including two local zero modes, namely, the rotational mode ($m = 0$) and the zero translational mode ($m = 1$). They appear due to the presence of two arbitrary parameters in solution (1): the position of the soliton center and the angle φ_0. For these modes, we can write a general formula in terms of u and v:

$$u_m^0 = r^{1-m}\left[\theta_0' - \frac{\sin\theta_0}{r}\right], \quad v_m^0 = r^{1-m}\left[\theta_0' + \frac{\sin\theta_0}{r}\right]. \quad (10)$$

Perturbations associated with φ_0 do not cause any displacement of the soliton, and the symmetry related to the displacement of the soliton center determines the presence of the zero translational mode. For us, it is important that the full set of modes of spectral problem (8) also includes nonzero modes associated with the soliton displacement. The coordinate of the soliton can be written in the form $\mathbf{X}(t) = (S/Na^2)\int(1 - \cos\theta)\,\mathbf{r}d^2x$ [12]. This allows us to determine the perturbations that lead to a finite velocity of the soliton $d\mathbf{X}(t)/dt$. For the case of small perturbations, with the use of Eqs. (5), we obtain

$$\frac{d\mathbf{X}}{dt} = \frac{Sl_0^2\omega_0}{Na^2}\int d^2x \cos\theta_0[(\sin\theta_0 \nabla\phi_0 + i\nabla\theta_0)\psi + \text{c.c.}].$$

Since $(\nabla\theta_0, \nabla\phi_0) \propto (\sin\chi, \cos\chi)$ and the other terms under the integral depend on r alone, only the modes with $m = \pm 1$ possess the property of interest. Thus, the dynamics of $\mathbf{X}(t)$ can only be associated with the modes characterized by $|m| = 1$. In addition, the motion of the soliton as a particlelike object without the excitation of magnon modes in the whole volume of the system is only possible when the corresponding mode is localized.

The magnon modes localized by the soliton were studied in [13, 6]. In [6], with the use of the two-parameter shooting method, we showed that a soliton with a relatively large radius possesses a set of local modes with $|m| \le m_{\max}(R)$. As the soliton radius decreases, the modes sequentially leave the gap region (the discrete spectrum region), pass to the continuous spectrum, and transform into quasilocal modes. For example, when $R \lesssim 2.8 l_0$, all of the modes with $|m| > 2$ cease being localized, and when $R \lesssim 1.5 l_0$, the system contains only one local nonzero mode with $m = -1$, which can be called the nonzero translational mode. As R decreases, the frequency of this mode $\omega_{m=-1}$ asymptotically approaches the boundary of the continuous spectrum while the exponential decrease in the magnetization perturbation away from the soliton becomes slower, $r_0 \gg l_0$. The wave function becomes delocalized, and the numerical analysis based on the two-parameter shooting method becomes difficult and unreliable (in [6], it was actually possible to consider only solitons with $R \gtrsim 0.3 l_0$). Therefore, the question of what should occur with a further decrease in the soliton radius remains open: whether the soliton will possess a nonzero translational mode in the case of small-radius solitons for the so-called magnetic skyrmions or this mode will pass to the magnon continuum by transforming into a quasilocal mode. As we noted above, this question is of fundamental significance in connection with the problem of applicability of Newtonian equations to the soliton dynamics.

The problem of the presence of a local mode can be solved independently by analyzing the scattering data. As was noted above, the local mode problem based on the two-parameter shooting method cannot be solved numerically for the limiting case of small soliton radius, but the problem of magnon scattering by a soliton can be numerically integrated for solitons as small as one likes with the use of the one-parameter shooting method [6]. The scattering data make it possible to derive a conclusion concerning the presence or absence of the local mode on the basis of the Levinson theorem, according to which the number of bound states and the total phase shift in the scattering data for a partial mode are related to each other. In the case of analyzing the soliton–magnon scattering, the problem is complicated by the fact that the effective potentials have singularities at zero (of the type of ν^2/r^2) and at infinity (μ^2/r^2), where the numbers ν and μ are not equal to m. Recently we generalized the Levinson theorem for such singular potentials [14] and showed that the total shift of the scattering phase (Eq. (9)) is determined by the expression

$$\delta_m(0) - \delta_m(\infty) = \pi\left[N_m^{\text{b}} + \frac{|\nu| - |\mu|}{2}\right].$$

In our case, $\nu = q - m$ and $\mu = q + m$ and the total phase shift can be represented as

$$\delta_m(0) - \delta_m(\infty) = \pi[N_m^b - \operatorname{sgn} m]. \qquad (11)$$

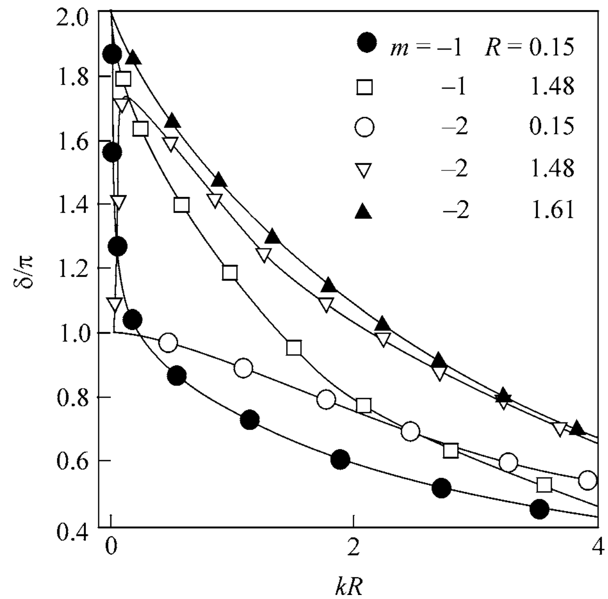
When the number of local modes changes by one, the total phase shift undergoes a jump by π, which can be used to determine the instant of disappearance of the local mode. The scattering data for the mode with $m = -1$ are shown in the figure. To illustrate the operation of the method, we present the scattering data for the mode with $m = -2$ in the same figure. In particular, as was shown in [6], a soliton with a sufficiently large radius has a local mode with $m = -2$. In this case, according to Eq. (11), the total phase shift is equal to 2π. In the figure, this situation corresponds to the curve marked with full triangles. As the soliton radius decreases, the local mode leaves the discrete spectrum region; hence, the number of bound states decreases by one and the total phase shift becomes equal to π (see the curves marked with empty triangles and empty circles). A numerical calculation performed for the mode $m = -1$ shows that, for any soliton radius, the total phase shift is equal to 2π (the curves with full circles and empty squares). A comparison of this result with Eq. (11) confirms the above conclusion that the nonzero local mode with $m = -1$ is always present for solitons of any radius.

To calculate the frequency of the nonzero translational mode, it is convenient to reformulate the spectral problem as a variational problem for the functional (Lagrangian)

$$\mathcal{L} = \omega_0 l_0^2(\mathbf{m}|\mathcal{H}|\mathbf{m}) - \omega_m(\mathbf{m}|\mathbf{m}). \qquad (12)$$

For a small-radius soliton, all higher modes with $|m| > 1$ cease being local. Among the remaining three modes with $m = -1$, 0, and +1, two are nonzero. One of the main points in using the variational approach is the adequate choice of the trial function. For this purpose, we use the closeness of the structure of soliton solution (3) to the Belavin–Polyakov solution in a wide range of coordinate values $r \ll r_0$, $r \geq R$. For the Belavin–Polyakov soliton, all three modes with $m = -1, 0, +1$ have nonzero frequencies and are determined by Eqs. (10) [15]. For small-radius solitons, the trial functions for the mode $m = -1$ are most naturally chosen in the form of combinations of the functions $u_{\pm1}^0$ and $v_{\pm1}^0$ given by Eqs. (10):

$$\begin{aligned} u_{-1}^{\omega} &= u_1^{\omega=0} + ar^2\theta_0' - br\sin\theta_0, \\ v_{-1}^{\omega} &= u_1^{\omega=0} + ar^2\theta_0' + br\sin\theta_0. \end{aligned} \qquad (13)$$



Phase shift for the modes with $m = -1$ and -2 and for various values of the soliton radius.

Effective Lagrangian (12) with these trial functions takes the form

$$L = (a+b)^2\left\{\langle\sin^2\theta_0\rangle - \frac{\omega}{2\omega_0}\langle 1-\cos\theta_0\rangle^2\right\}$$

$$+ \frac{4ab\omega}{\omega_0 l_0^2}\langle r^2(1-\cos\theta_0)\rangle + 2a^2\langle r^2\theta_0'^2 - \sin^2\theta_0\rangle,$$

where $\langle f(r)\rangle = \int_0^\infty f(r)\,rdr$. The conditions $\partial L/\partial a = 0$ and $\partial L/\partial b = 0$ lead to the following expression for the eigenfrequency of the nonzero translational mode:

$$\omega_{m=-1} = \frac{2\omega_0 l_0^2 F}{4\langle r^2(1-\cos\theta_0)\rangle + l_0^2 F\langle 1-\cos\theta_0\rangle^2}, \qquad (14)$$

where $F = \langle\sin^2\theta_0\rangle\langle r^2\theta_0'^2 - \sin^2\theta_0\rangle$.

For the small-radius soliton of interest, the core structure is close to the structure of the Belavin–Polyakov soliton and, for $r \ll r_0$, the relation $r^2\theta_0'^2 = \sin^2\theta_0$ is satisfied. Therefore, the quantity F is determined by the region lying away from the center, $r \geq r_0$:

$$F \approx \frac{\omega_0 - \Omega}{\omega_0 l_0^2}\langle\sin^2\theta_0\rangle\langle r^2\sin^2\theta_0\rangle.$$

This quantity is always small. Now, estimating the average values using approximate solution (3), we obtain an expression for the eigenfrequency (14):

$$\omega_{m=-1} \approx (\omega_0 - \Omega)\left[1 - \frac{2R^2(\omega_0 - \Omega)}{\omega_0 l_0^2}\ln\frac{\omega_0}{\omega_0 - \Omega}\right]. \quad (15)$$

Hence, one can see that, when the soliton radius decreases, the mode with $m = -1$ remains local but, as $R \longrightarrow 0$, its frequency asymptotically approaches the boundary of the continuum.

Thus, based on the analysis of the scattering problem for the generalized Schrödinger equation describing magnons on the background of a soliton, we proved the existence of a local translational magnon mode with nonzero frequency and azimuthal number $m = -1$. We analytically determined the frequency of this mode, which asymptotically tends to the spectrum boundary ω_0 as the soliton radius decreases $R \ll l_0$. In terms of the soliton coordinate, this mode corresponds to the Larmor precession of the soliton center with a small amplitude. The inclusion of this mode and the mode with $m = +1$, which acquires a nonzero frequency in the presence of fixed boundary conditions for a circular magnet of radius L (this leads to the appearance of the restoring force $F_{\text{ext}} = -\kappa\mathbf{r}$, where $\kappa \propto \exp(-L/r_0)$), unambiguously fits the Newtonian soliton dynamics described by Eq. (4). In this case, a slow motion of the soliton occurs with the mode frequency $\omega_{m=1}$, which, for small values of κ, can be represented as κ/G and contains no inertial mass, as well as Larmor precession with the nonzero mode frequency $\omega_{m=-1} = G/m_*$. The calculation performed for the frequency $\omega_{m=-1}$ allows us to determine the effective soliton mass

$$m_* = \frac{G}{\omega_{m=-1}} \approx \frac{4\pi\hbar S}{a^2(\omega_0 - \omega)}. \quad (16)$$

As the soliton radius decreases, $\omega \longrightarrow \omega_0$ and the soliton mass infinitely increases. Hence, the soliton loses its mobility with decreasing radius.

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REFERENCES

1. A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, *Nonlinear Waves of Magnetization: Dynamical and Topological Solitons* (Naukova Dumka, Kiev, 1983) [in Russian]; Phys. Rep. **194**, 117 (1990); V. G. Bar’yakhtar and B. A. Ivanov, Sov. Sci. Rev., Sect. A **16**, 3 (1993).
2. A. A. Belavin and A. M. Polyakov, Pis’ma Zh. Éksp. Teor. Fiz. **22**, 503 (1975) [JETP Lett. **22**, 245 (1975)].
3. R. H. Hobard, Proc. Phys. Soc. **82**, 201 (1963); G. H. Derrick, J. Math. Phys. **5**, 1252 (1964).
4. B. A. Ivanov and A. M. Kosevich, Pis’ma Zh. Éksp. Teor. Fiz. **24**, 495 (1976) [JETP Lett. **24**, 454 (1976)].
5. A. S. Kovalev, A. M. Kosevich, and K. V. Maslov, Pis’ma Zh. Éksp. Teor. Fiz. **30**, 321 (1979) [JETP Lett. **30**, 296 (1979)].
6. D. D. Sheka, B. A. Ivanov, and F. G. Mertens, Phys. Rev. B **64**, 024 432 (2001).
7. F. G. Mertens and A. R. Bishop, in *Nonlinear Science at the Dawn of the 21st Century*, Ed. by P. L. Christiansen, M. P. Soerensen, and A. C. Scott (Springer, Berlin, 2000).
8. N. Papanicolaou and T. N. Tomaras, Nucl. Phys. B **360**, 425 (1991); N. Papanicolaou and W. J. Zakrzewski, Physica D (Amsterdam) **80**, 225 (1995); B. Piette and W. J. Zakrzewski, Physica D (Amsterdam) **119**, 314 (1998).
9. K. Lonngren and A. Scott, *Solitons in Action* (Academic, New York, 1978).
10. B. A. Ivanov, H. J. Schnitzer, F. G. Mertens, and G. M. Wysin, Phys. Rev. B **58**, 8464 (1998).
11. D. D. Sheka, I. A. Yastremsky, B. A. Ivanov, *et al.*, Phys. Rev. B **69**, 054 429 (2004).
12. D. D. Sheka, C. Schuster, B. A. Ivanov, and F. G. Mertens, cond-mat/0505542 (2005).
13. F. K. Abdullaev, R. M. Galimzyanov, and A. S. Kirakosyan, Phys. Rev. B **60**, 6552 (1999).
14. D. Sheka, B. Ivanov, and F. G. Mertens, Phys. Rev. A **68**, 012 707 (2003).
15. B. A. Ivanov, Pis’ma Zh. Éksp. Teor. Fiz. **61**, 898 (1995) [JETP Lett. **61**, 917 (1995)].

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