Soliton–magnon scattering in a two-dimensional isotropic magnetic material

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We use the generalized σ-model to analytically study the solution of the problem of magnon scattering in two-dimensional isotropic ferromagnets and antiferromagnets in the presence of a Belavin–Polyakov soliton. We obtain the exact analytical solution to this problem for the partial mode with the azimuthal quantum number $m=1$. The scattering amplitude for other values of m (i.e., values not equal to unity) are studied analytically in the long- and short-wavelength approximations and also numerically for an arbitrary value of the wave number. We establish the general laws governing the soliton–magnon interaction. For a magnetic material of finite dimensions we calculate the frequencies of the magnon modes. We also use the data on local modes to derive the equations of motion of the soliton. Finally, we calculate the low-temperature (long-wavelength) asymptotic behavior of the magnon density of states due to the soliton–magnon interaction. [S1063-7761(99)02309-4]

1. INTRODUCTION

It is now firmly established that solitons play an important role in low-dimensional magnetism, i.e., in one-dimensional (1D) and two-dimensional (2D) magnetic materials. Studies began with the simpler 1D case. Krumhansl and Schrieffer[1] found that solitons (kinks) must be considered on an equal basis with magnons as elementary excitations in the derivation of the thermodynamics of 1D magnetic materials. Currie *et al.*[2] construct a consistent phenomenological theory of solitons, in which a nontrivial fact was established, namely, that the kink–magnon interaction substantially alters the magnon density of states, which has an effect on the thermodynamic properties of the system. In particular, the temperature dependence of the soliton density is determined by the shift in the magnon phase in kink–magnon scattering and can vary substantially for magnetic materials with different kink–magnon interactions.[3,4]

A special role in soliton phenomenology is assigned to local magnon modes, which are spin waves localized at a magnetic soliton. For instance, the number of such modes determines the total variation of the magnon density of states and hence the temperature dependence of the kink density.[3] More than that, local modes are interesting objects by themselves, and their study is linked to direct experiments in exciting and detecting them, since by characterizing the intrinsic latent degrees of freedom of the soliton the local modes are the cause of soliton magnetic resonance at the characteristic frequencies of ''intrinsic'' motion.[5]

Important results in the soliton thermodynamics of 2D magnetic materials were obtained by Mertens *et al.*[6] and in the research that followed (see the review articles in Refs. 3 and 7). In research devoted to 2D solitons, the density of solitons (vortices) was taken as an external parameter of the theory. This approach was also used in analyzing the data of the experiments in which the contribution of localized 2D solitons to the relaxation of spin excitations was observed.[8–14] The main difficulty in analyzing 2D systems lies in the absence of exact analytical solutions for most models. Usually the solitons are treated numerically by diagonalizing with respect to small discrete systems.[15–18] In such finite geometry the soliton–magnon interaction manifests itself primarily in the existence of specific Goldstone local modes with anomalously low frequencies and in the excitation of magnon modes by soliton motion. Thanks to the reverse effect, it was possible to describe the dynamical parameters of a soliton by the data on local modes.[19]

In this connection, an important role is played by the analysis of such 2D models for which analytical results can be obtained and the general laws governing the soliton–magnon interaction can be established. Only one exact analytical solution of this type is known, the Belavin–Polyakov (BP) soliton, which describes a topological soliton in an isotropic 2D magnetic material.[20] The existence of local modes in such a system was predicted in Ref. 21 for an isotropic 2D ferromagnet and in Ref. 22 for an antiferromagnet. In particular, it was found that a BP soliton with a topological charge ν has $2|\nu|$ local modes of zero frequency (local zero-frequency modes).

In the present paper we construct a solution of the problem of scattering of magnons by a BP soliton in 2D magnetic materials. In Sec. 2 we examine the generalized σ-model, which can be used to describe ferromagnets and antiferromagnets, as well as ferrimagnets near the point of compensation of the sublattice spins. In Sec. 3 we formulate the scattering problem for this model and obtain its exact solution for the partial mode with azimuthal quantum number $m=1$. Sections 4 and 5 are devoted to calculations of the scattering amplitude for the other values of $m(n\neq 1)$ analytically in the long-wavelength approximation $kR\ll 1$ (Sec. 4) and numerically for arbitrary values of kR (Sec. 5), where k is the wave number and R is the radius of the soliton core. In the sections that follow we use the results to describe the various physical properties of solitons and local magnon modes. Section 6 deals with calculations of the frequency of the magnon modes for a magnetic material of finite dimensions. In the same section, using the data on local modes, we derive the equations of soliton motion. In Sec. 7 we calculate the magnon density of states for which the soliton–magnon interaction is responsible. In the Conclusion we discuss the different ways in which the theory could develop and the possible applications.

2. THE MODEL. ELEMENTARY EXCITATIONS

A broad class of classical isotropic Heisenberg 2D magnetic materials can be described dynamically in terms of the classical unit vector $\mathbf{n}$ of the order parameter, i.e., $n_z=\cos\theta$ and $n_x+in_y=\sin\theta\exp\{i\phi\}$. The dynamics of a classical ferromagnet is described by the Landau–Lifshitz equation for the normalized magnetization,[23] which acts as the dynamic variable $\mathbf{n}$. In a classical antiferromagnet, the dynamic variable is the antiferromagnetism vector, which in the long-wavelength approximation can be assumed to be a unit vector. The dynamics of an antiferromagnet is described by the equations of the σ-model of the $\mathbf{n}$-field.[24,25]

In the interests of generality we examine two types of magnetic materials within a unified approach, more precisely, on the basis of a generalized σ-model, whose Lagrangian in the 2D case can be written[26]

$$L=\frac{A}{2}\int d^2x\left\{\frac{1}{c^2}\left(\frac{\partial\theta}{\partial t}\right)^2-(\nabla\theta)^2+\sin^2\theta\left[\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2-(\nabla\phi)^2\right]-\frac{2}{D}(1-\cos\theta)\frac{\partial\phi}{\partial t}\right\}, \tag{1}$$

where $A=JS^2$, where J is the exchange integral and S is the atomic spin. The specific type of magnetic material is determined by the relationship between the parameters c and D. To describe a ferromagnet we must drop the second time derivatives in the equations of motion, i.e., formally let c go to infinity. The dynamic term in the Lagrangian of the ferromagnet is of a purely gyroscopic nature, with the parameter D having the meaning of the spin stiffness of the ferromagnet. The dynamics of an isotopic σ-model describing an antiferromagnet has a Lorentz-invariant form with a characteristic speed parameter c. For an antiferromagnet there is no gyroscopic term (the coefficient D can be taken to infinity). Note that the generalized σ-model for finite D and c describes a ferrimagnet near the point at which the mechanical moments of the sublattices are balanced. For such a magnetic material the gyroscopic term has the same structure as in a ferromagnet but is proportional to the small parameter $(S_1-S_2)/(S_1+S_2)$, where S_1 and S_2 are the average mechanical moments of the sublattices.[27,28]

The simplest elementary linear excitations of a 2D isotropic magnetic material that arise against the background of the ground homogeneous state are the magnons belonging to the continuous spectrum. If we select the orientation of the order-parameter vector $\mathbf{n}$ along the polar axis, we get magnon solutions in the form of a circularly polarized wave $\theta=\mathrm{const}\ll 1$, $\Phi=\mathbf{kr}-\omega(k)t$. The dispersion law for a ferromagnet is quadratic, $\omega_{FM}(k)=Dk^2$. For an antiferromagnet the dispersion law is linear, $|\omega_{AFM}(k)|=ck$, and there are two degenerate branches with opposite circular polarizations, $\omega=\pm ck$, which is equivalent to the possibility of linear polarization of magnons.

The simplest static nonlinear excitations in the 2D case are the BP solitons,[20]

$$\tan\frac{\theta_0}{2}=x^{-|\nu|},\quad \phi_0=\varphi_0+\nu\chi,\quad x=\frac{r}{R}, \tag{2}$$

which, naturally, has the same form for a ferromagnet, an antiferromagnet, and a ferrimagnet. Here r and χ are the polar coordinates in the plane of the magnetic material, the integer ν is the topological charge of the soliton, and R and φ_0 are arbitrary parameters.

The energy of such a soliton is given by the formula

$$E_0=4\pi A|\nu| \tag{3}$$

and is independent of R and φ_0. The ambiguity in the choice of φ_0 is a characteristic feature of many models and a consequence of the isotropy of the Heisenberg exchange. The existence of an arbitrary parameter R (the soliton radius) and the fact that the energy is independent of R are related to the scale invariance of the static two-dimensional σ-model.[23] Obviously, this symmetry is broken in dynamics, with the exception of the trivial case of a pure antiferromagnet and translational motion, when everything reduces to Lorentz transformations.

In analyzing the static solutions it is convenient to introduce the complex-valued order parameter $w=(n_x+in_y)/(1-n_z)$ and interpret it as a function of the complex variable $\zeta=re^{i\chi}$ describing the position of a point in the plane of the magnetic material. In terms of these variables, the static equations of the σ-model reduce to the self-duality equation[29] $\partial w/\partial\zeta=0$ or $\partial w/\partial\overline{\zeta}=0$. The BP soliton corresponds to the simplest solution of this equation of the form

$$w_0=A\zeta^{\nu}\ \text{for}\ \nu>0,\quad w_0=A\overline{\zeta}^{-\nu}\ \text{for}\ \nu<0. \tag{4}$$

There are also more general solutions to this equation of the form $=f(\zeta)$ or $w=f(\overline{\zeta})$, where f is any analytic function of the complex variable ζ. In particular, the static multisoliton solution with the topological charge ν depends on $2|\nu|$ parameters[23] and can be written

$$w=A\frac{\Pi_{k=1}^{\nu}(\zeta-a_k)}{\Pi_{k=1}^{\nu-1}(1-b_k\zeta)}\quad \text{for} \quad \nu>0 \tag{5}$$

(constructing the same general solution for $\nu<0$ is a trivial task). The energy associated with this solution is determined by (3) and is independent of the parameters A, a_k, and b_k. We associate the soliton center with the value $\theta=\pi$, this solution has (for different a_k and b_k) ν solitons with unit topological charges at the points $\zeta=a_k$. If all a_k coincide, then at $b_k=0$ the solution (5) coincides with (4) and describes one soliton with the topological charge ν at point $\zeta=a_k$. Thus, variations in the parameters a_k and b_k has a strong effect on the structure of the soliton but do not change its energy or topological charge. This means that a BP soliton has extremely high internal degeneracy, which reflects the property of conformal invariance of the static two-dimensional σ-model.[20,29] Hence a BP soliton consists of a set of local modes with a zero frequency. The explicit form of these zero-frequency modes can be obtained by varying (5) in the parameters a_k and b_k. In the limit $a_k, b_k\to 0$, the soliton can be represented by the expansion

$$\Omega\equiv\frac{w-w_0}{w_0}=\sum_{m=-\nu+1}^{\nu}\frac{A_m}{\zeta^m}\quad \text{for} \quad A_m\to 0, \tag{6a}$$

or, introducing the deviations θ and ϕ from the quantities θ_0 and ϕ_0 into the simplest equation (2), by the formula

$$\theta-\theta_0+i\sin\theta_0(\phi-\phi_0)=-\frac{\sin\theta_0 A_m}{(\overline{\zeta})^m}. \tag{6b}$$

This implies that there are $2|\nu|$ independent types of small perturbations that do not alter the soliton energy. Their form is determined by the function $\Omega\propto(\overline{\zeta})^{-m}\propto\exp\{im\chi\}$. This is equivalent to the statement that $2|\nu|$ local modes with a zero frequency are associated with a BP soliton (see below).

3. MAGNON MODES IN THE PRESENCE OF A SOLITON

To describe the magnon excitations that arise against the background of a BP soliton, it is convenient to introduce local coordinates $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ characterizing the distribution of the order parameter in a fixed soliton: $\mathbf{e}_3$ coincides with the order parameter $\mathbf{n}_0$ of the immobile soliton (2), $\mathbf{e}_1=\mathbf{e}_y\cos\phi_0-\mathbf{e}_x\sin\phi_0$, and $\mathbf{e}_2=\mathbf{e}_3\times\mathbf{e}_1$. Then the linear oscillations of the order parameter can be described in terms of the projections of $\mathbf{n}$ on the local axes $\mathbf{e}_1$ and $\mathbf{e}_2$: $\vartheta=\mathbf{n}\cdot\mathbf{e}_1$ and $\mu=\mathbf{n}\cdot\mathbf{e}_2$ (ϑ and $\mu/\sin\theta_0$ are the small deviations from θ_0 and ϕ_0, respectively).

The linearized equations for ϑ and μ can be represented in the form of the system of equations

$$\left[-\nabla_x^2+\frac{1}{x^2}\frac{\partial^2}{\partial\chi^2}+U_1(x)\right]\vartheta+\frac{2\nu}{x^2}\times\cos\theta_0\frac{\partial\mu}{\partial\chi}+\frac{R^2}{c^2}\frac{\partial^2\vartheta}{\partial t^2}+\frac{R^2}{D}\frac{\partial\mu}{\partial t}=0,$$

$$\left[-\nabla_x^2+\frac{1}{x^2}\frac{\partial^2}{\partial\chi^2}+U_2(x)\right]\mu-\frac{2\nu}{x^2}\times\cos\theta_0\frac{\partial\vartheta}{\partial\chi}+\frac{R^2}{c^2}\frac{\partial^2\mu}{\partial t^2}-\frac{R^2}{D}\frac{\partial\vartheta}{\partial t}=0, \tag{7}$$

where $\nabla_x^2\equiv(1/x)\partial/\partial x(x\partial/\partial x)$ is the radial part of the Laplace operator and $U_1(x)=(\nu/x)^2\cos 2\theta_0$ and $U_2(x)=\cot\theta_0\nabla_x^2\theta_0-(d\theta_0/dx)^2$ are the "potentials."[18,19] Using the explicit form (2) for the static solution, we can easily show that the "potentials" in both equations are the same. This fact is unique for the isotropic σ-model. For instance, for vortices in a magnetic material with easy-magnetization planes,[18,19] the potentials differ substantially. The very fact that the potentials are different not only complicates the analysis technically (in comparison to the ordinary Schrödinger equation) but also introduces serious problems. In particular, for systems of the form (7) with unequal potentials U_1 and U_2 many general assertions of the type of the oscillation theorem have yet to be formulated. In Ref. 18 it was shown that equations of this form may have truly localized states with an exponential decrease of the wave function and energies inside the continuous spectrum, which is forbidden for equations of the Schrödinger form.

In the degenerate case considered here the magnon modes can be described by a single complex-valued parameter $\Psi=\vartheta+i\mu$, which obeys the second equation

$$\left[-\nabla_x^2+\frac{1}{x^2}\frac{\partial^2}{\partial\chi^2}+\frac{\nu^2}{x^2}\cos 2\theta_0\right]\Psi-i\frac{2\nu}{x^2}\cos\theta_0\frac{\partial\Psi}{\partial\chi}+\frac{R^2}{c^2}\frac{\partial^2\Psi}{\partial t^2}-i\frac{R^2}{D}\frac{\partial\Psi}{\partial t}=0, \tag{8}$$

whose analysis is almost the same as that of the Schrödinger equation. It is convenient to seek the solution of Eq. (8) in the form of a partial-wave expansion:

$$\Psi=\sum_{m=-\infty}^{\infty}f_m\exp\{im\chi-i\omega t\}. \tag{9}$$

Here each partial wave f_n is an eigenfunction of the spectral problem

$$\hat{H}f_m=\varkappa^2 f_m,\quad \varkappa=kR, \tag{10}$$

for the 2D radial Schrödinger operator $\hat{H}=-\nabla_x^2+U_m(x)$ with the potential

$$U_m(x)=\frac{m^2+2m\nu\cos\theta_0+\nu^2\cos 2\theta_0}{x^2}.$$

The spectrum of the problem (10) is continuous and is described by functions of the form $f_m^{\varkappa}$, with $\varkappa\geqslant 0$. Clearly, the zero-frequency modes f_m^0 correspond to solutions[21]

$$f_m^{(0)}=x^{-m}\sin\theta_0. \tag{11}$$

These modes correspond to perturbations of the form (6), i.e., their presence is due to the conformal invariance of the problem. Here and below, for the sake of definiteness, we examine the case where $\nu>0$, and to analyze solitons with $\nu<0$ it is enough to replace m by $-m$. This solution behaves regularly as $r\to 0$ only for partial modes with $-\infty < m \leq \nu$. A simple analysis of Eq. (11) shows that for $-\nu + 1 \leq m < \infty$ the function $f_m^{(0)}$, regular as $r\to 0$, also decreases far from the soliton. Hence for $-\nu+1\leq m\leq \nu$ these functions are finite over the entire range of r. This corresponds to the earlier conclusion that a BP soliton with a topological charge ν has $2|\nu|$ local modes represented in (6). Note that the physical meaning of two of these modes is obvious: the translational mode $f_{m=1}^{(0)}$ describes the displacement of the soliton as a whole, and the rovibrational mode $f_{m=0}^{(0)}$ describes the rotation and change of the soliton radius (which corresponds to an ambiguity in the choice of the position of the soliton center and to arbitrary values of φ_0 and R, respectively). The established bound states (local modes) are limits for the magnon modes of the continuous spectrum as $\varkappa\to 0$, on contrast to the case of 1D magnetic materials (see the review in Ref. 3).

Using the standard method of varying the arbitrary constant, we can find the second linearly independent solution of Eq. (10) with $\varkappa=0$:

$$f_m^{(1)}=x^m\left(\frac{x^{2\nu}}{m+\nu}+\frac{2}{m}+\frac{x^{-2\nu}}{m-\nu}\right)\sin\theta_0, \tag{12}$$

which is regular at zero when $m>\nu$.

Thus, at $\omega=0$ one of the solutions, (11) or (12), for all m has no singularities at zero. We use this solution to analyze scattering at small (but finite) values of ω in the range of small r.

The exact solutions $f_m^{(0)}$ that have been found can be used to simplify the problem of the analysis of the continuous spectrum on the basis of the Darboux transformation.[30] The same approach has been used in Refs. 31 to study the 1D case. To explain the method, we introduce the Hermitian-conjugate lowering and raising operators

$$\hat{A}=-\frac{d}{dx}+\frac{f_m^{(0)\prime}}{f_m^{(0)}},\quad \hat{A}^\dagger=\frac{d}{dx}+\frac{1}{x}+\frac{f_m^{(0)\prime}}{f_m^{(0)}}$$

such that $\hat{A}f_m^{(0)}=0$ (here and in what follows a prime stands for a derivative with respect to x). By introducing these operators we can represent the Schrödinger operator $\hat{H}$ in the factorized form $\hat{H}=\hat{A}^\dagger\hat{A}$. What is important is that this makes it possible to reformulate the initial problem (10) in terms of the eigenfunctions $g_m^\varkappa=\hat{A}f_m^\varkappa$ of the spectral problem of the form

$$\hat{\mathscr{H}}g_m=\varkappa^2 g_m,\quad \hat{\mathscr{H}}\equiv\hat{A}\hat{A}^\dagger=-\nabla_x^2+\mathscr{U}_m, \tag{13}$$

where the potential is

$$\mathscr{U}_m(x)=\frac{(m-1)^2+\nu^2+2\nu(m-1)\cos\theta_0}{x^2}.$$

Note that far from the soliton (as $\theta_0\to 0$) the potential $\mathscr{U}_m$ becomes the centrifugal potential of the form $(\nu+m-1)^2/r^2$, which depends explicitly on the azimuthal number labeled $m-1$, which explains the terminology used for the operators $\hat{A}$ and $\hat{A}^\dagger$.

The initial function f_m is restored by applying the raising operator:

$$f_m^\varkappa=\frac{1}{\varkappa^2}\hat{A}^\dagger g_m^\varkappa. \tag{14}$$

The transformation we have just carried out simplifies the problem for the translational mode ($m=1$) substantially. Indeed, in this case $\mathscr{U}_1(x)=\nu^2/x^2$ for all values of x, i.e., $g_1^\varkappa$ determines the free motion. In view of this, the regular solution has the form $g_1^\varkappa=-J_\nu(\varkappa x)$. Restoring the initial function by (14), we get

$$f_1^\varkappa=J_{\nu+1}(kr)-\frac{2\nu}{kr}\frac{J_\nu(kr)}{(r/R)^{2\nu}+1}. \tag{15}$$

The existence of this exact solution for all values of the wave vector $\mathbf{k}$ is a unique property of the model (1). For other values of m (as earlier, for the case of magnon scattering by magnetic vortices in ferromagnets[19] and antiferromagnets[18] with planes of easy magnetization), the problem can be solved only approximately or by numerical methods.

The solution (15) demonstrates an important feature of magnon modes, which is absent in the cases discussed in Refs. 18 and 19, where the exponential decrease of the deviation of magnetization from the easy-magnetization plane in a vortex far from the vortex center is a characteristic feature. Equation (15) shows that the deviation of $f_1^\varkappa$ from the asymptotic term $J_{\nu+1}(kr)$ is not localized in a region with a definite radius; instead it is characterized by a slower (power-law) decrease. More than that, for the most interesting cases of long-wavelength asymptotic behavior, for $k\ll 1/R$, the solution (15) over a broad range of values of r, or $R\ll r\ll 1/k$, has the same form as a combination of Bessel and Neumann functions, $J_{\nu+1}(z)\propto z^{\nu+1}$ and $N_{\nu+1}\propto z^{-(\nu+1)}$, i.e., the second term in (15) imitates the presence of the function N. (Below we will see that this property remains valid for all values of m.)

For magnetic vortices, the corresponding corrections are exponentially decreasing functions of the form $\exp\{-r/r_v\}$, where r_v is the radius of the vortex core. In view of this, the scattering amplitude, i.e., the coefficient of the Neumann function, can be determined from the coefficient of $1/z^m$ in the region where $z\ll 1$. This is not true, however, in our case, with the result that the method developed in Ref. 19 for magnetic vortices and used to analyze the scattering matrix by analyzing the corrections to the zero-frequency modes in the region $r_v\ll r\ll 1/k$ needs to be thoroughly modified[1)] if we wish to use it in our problem. This modification is done in Sec. 4.

On the other hand, the terms with a power-law decrease of the form $1/r^a$, $a>1/2$, must be taken into account when we describe the properties of magnon modes in a magnetic material of finite size with a soliton. This is done in Sec. 6.

4. SCATTERING IN THE LONG-WAVELENGTH LIMIT

To describe the scattering of magnons by a BP soliton, we note that free magnon states can be found if we set $\nu=0$ in the "potential" $U_m(x)$. The resulting magnon modes $f_{m,\nu=0}^\varkappa=J_m(z)$, with $z=kr$, are the partial cylinder waves of a plane spin wave of the form

$$\exp\{i\mathbf{k}\cdot\mathbf{r}-i\omega t\}=\sum_{m=-\infty}^{\infty} i^m J_m(z)\exp\{im\chi-i\omega t\}. \quad (16)$$

In the presence of a soliton the behavior of the magnon solutions can be analyzed at large distances from the soliton ($r\gg R$). In view of the asymptotic behavior $U_m(x)\approx n^2/x^2$, in the leading approximation in $1/x$ we have the usual result[21]

$$g_m\propto G_m(z)\equiv J_{|n|}(z)+\sigma_m^\nu N_{|n|}(z), \quad n=\nu+m-1,$$

$$f_m\propto F_m(z)\equiv J_{|p|}(z)+\sigma_m^\nu N_{|p|}(z), \quad p=\nu+m \quad (17)$$

(below we also use the notation involving n and p, and $G_m(z)$ and $F_m(z)$ for the combination of cylinder functions of the specific form presented in (17) with allowance for σ). A comparison of the asymptotic behaviors of $G_m(z)$ and $F_m(z)$ with each other and with the solution (16) for free magnons suggests that $\sigma_m^\nu\equiv\sigma_m^\nu(\varkappa)$ determines the soliton–magnon scattering amplitude. Since the coefficients σ are the same for F_m and G_m, to calculate the scattering amplitude $\sigma_m(\varkappa)$ we can use the initial problem or the modified problem. In particular, there is no scattering for the translational mode. Unfortunately, there is no way in which we can find analytical solutions for the other modes, but the scattering can be analyzed fairly thoroughly in the limiting cases.

To analyze soliton–magnon scattering in the case of small k, we can use the fact that at $k=0$ we know the exact solutions f_m^0: (11) for $m\leqslant\nu$ and (12) for $m>\nu$. In this case, we can construct the solution for small but finite $k(k\ll 1/R)$ by using a perturbation-theory expansion in k^2. To this end we seek the solution in the form $f_m^\varkappa=f_m^0(1+\varkappa^2\alpha(x))$, where $\varkappa^2\alpha(x)\ll 1$. The function $\alpha(x)$ is determined by an inhomogeneous second-order linear equation, whose solution can be found by the method of variation of the arbitrary constant if the two linearly independent solutions of the homogeneous problem are known. For a magnetic material with easy-magnetization planes this can be done only for the translational mode.[19] In the case of an isotropic magnetic material the solutions can be found by this method for arbitrary values of m (see Refs. 32 and 22).

In deriving a specific solution it is convenient to employ the modified problem by using the first-order equation $\hat{A}^\dagger g_m^\varkappa=\varkappa^2 f_m^0$, where f_m^0 is the zeroth solution, bounded as $x\to 0$. When $m\leqslant\nu$, the function $f_m^{(0)}$ is such a solution, from which we easily find that

$$g_m^\varkappa=\frac{\varkappa^2}{x f_m^{(0)}}\Phi^{(0)}(x),$$

$$\Phi^{(0)}(x)=\int_0^x (f_m^{(0)}(\xi))^2\xi d\xi, \quad \text{for } m\leqslant\nu. \quad (18a)$$

The same formulas can be used to readily restore the explicit form of the solution $f_m^\varkappa$ of the initial problem:

$$f_m^\varkappa(x)=f_m^{(0)}(x)\left[1-\int_0^x\frac{g_m^\varkappa(\xi)}{f_m^{(0)}(\xi)}d\xi\right]. \quad (18b)$$

Analysis of this solution has shown that in a broad interval of r values, $0\leqslant r\ll R^s(1/k)^{1-s}$ (the values of parameter s are between 0 and 1 and depend on m), the addition to the zeroth solution $f_m^{(0)}$ is small and perturbation-theory techniques can be used.

The same laws stand for the magnon mode with $|m|=1$ scattered by a vortex in a magnetic material with easy-magnetization planes.[19] Since the deviations from the asymptotic solution were found to be exponentially small, both solutions are valid for $R\ll r\ll 1/k$, the asymptotic solution (18b) and of the form (17). This made it possible to find the coefficient of the Neumann function $N_{|n|}(kr)$ (with allowance for the fact that $N_{|n|}(kr)\propto(kr)^{-|n|}$ where $kr\ll 1$) and to write an analytical formula for $\sigma_{|m|=1}(k)$. In our case, however, the situation is more complicated. As noted in Sec. 3 in the discussion concerning the exact solution (15), the asymptotic solutions far from the soliton contains corrections that decrease by a power law. Although they do decrease faster than the asymptotic solution (18b), it is very important to account for them. In particular, they may have the same form as the Neumann function for $z\ll 1$.

Thus, to calculate the scattering amplitude we must compare the approximate solution (18b) not with the asymptotic form (17) but with the refined solution that allows for terms increasing by a power law far from the soliton. For $m\neq 1$ the corrections can be expressed by exact formulas, but they can easily be calculated in the long-wavelength approximation $k\ll 1/R$, where we can assume that $kR\ll z=kr\ll 1$.

To do this, we introduce the variable $z=kr=\varkappa x$ into Eq. (13). Then the combination $(R/r)^\nu$ in $\theta_0(r)$ becomes $\varkappa^\nu/z^\nu$ and vanishes for finite z as $\varkappa\to 0$. Hence in the limit $\varkappa=kR\to 0$ Eq. (13) simply becomes a Bessel equation with the solution (17), and the corrections can be found by a series expansion in powers of $(\varkappa/z)^\nu$. Keeping only the first nonvanishing approximation in $\varkappa^\nu$ and representing the asymptotic solution in the form $g_m^\varkappa(z)=G_m(z)+\mathscr{G}_{\nu,m}^\varkappa(z)$, we arrive at the inhomogeneous Bessel equation

$$\nabla_z^2\mathscr{G}+\left(1-\frac{n^2}{z^2}\right)\mathscr{G}=\frac{4\nu(1-m)}{z^2}\left(\frac{\varkappa}{2}\right)^{2\nu}G_m.$$

We see that to this accuracy the solution far from the soliton can be expressed in terms of the universal function $\mathscr{G}_{|n|\nu}(z)$,

$$g_m^\varkappa(z)=G_m(z)+4\nu(m-1)\varkappa^{2\nu}\mathscr{G}_{|n|,\nu}(z), \quad (19)$$

which can be found by solving an equation of the form

$$\nabla_z^2\mathscr{G}_{|n|,\nu}+\left(1-\frac{n^2}{z^2}\right)\mathscr{G}_{|n|,\nu}=-\frac{G_m}{z^{2(\nu+1)}}.$$

Using the standard method of variation of an arbitrary constant, we can write the solution of this equation in integral form:

$$\mathscr{G}_{|n|,\nu}(z)=\frac{\pi}{2}N_{|n|}\int_z^\infty\frac{G_m(z)J_{|n|}(z)}{z^{2(\nu+1)}}dz - \frac{\pi}{2}J_{|n|}\int_z^\infty\frac{G_m(z)N_{|n|}(z)}{z^{2(\nu+1)}}dz. \quad (20)$$

Here integration can be carried out exactly, and at $|n|=\nu$ the answer is $\mathscr{G}_{\nu,\nu}=-g_\nu^{(0)}/8\nu^2 z^{2\nu}$.

But if $|n|\neq\nu$, we have the recurrence relation

$$\mathcal{G}_{|n|,\nu}=\mathcal{G}_{\nu-1}A_{|n|,\nu}+B_{|n|,\nu},$$

$$A_{|n|,\nu}\equiv\frac{2\nu-1}{2\nu(n^2-\nu^2)},\quad B_{|n|,\nu}\equiv\frac{zg_{n-1}^{(0)}+(\nu-n)g_n^{(0)}}{4\nu(n^2-\nu^2)z^{2\nu}},$$

which yields

$$\mathcal{G}_{|n|,\nu}=\mathcal{G}_1\prod_{k=2}^{\nu}A_{|n|,k}+\sum_{k=2}^{\nu-1}B_{|n|,k}\prod_{i=k+1}^{\nu}A_{|n|,i}+B_{|n|,\nu}.$$

Limiting ourselves to corrections to the Bessel function, i.e., taking $G_m=J_{|n|}$ in (20), we arrive (after involved calculations that use the properties of cylinder functions) at an expression for $\mathcal{G}_1$:

$$\mathcal{G}_{|n|,1}(z)=-\frac{J_{|n|}(z)}{4(n+1)}\left[\frac{1}{z^2}+\frac{\ln(z/2)-\psi(|n|+1)}{n(n+1)}\right]+\frac{J_{|n|-1}(z)}{4z(n^2-1)}+\frac{1}{4|n|(n^2-1)}\times\sum_{k=1}^{\infty}(-1)^k\frac{(|n|+2k)J_{|n|+2k}}{k(|n|+k)}+\frac{\pi N_{|n|}(z)}{8|n|(n^2-1)},\tag{21}$$

where $\psi(x)$ is the Euler psi function.

Thus, as with the exact solution (15), the asymptotic behavior of the solution for $r\gg R$ differs from that in magnetic materials with easy-magnetization planes discussed earlier. Even if $\sigma=0$, i.e., there is no scattering, in the region far from the solution but with r finite $(R\ll r\ll 1/k)$, the solution contains a number of terms that formally diverge as $z\to 0$ $(kr\ll 1)$. In this case, for $\sigma\neq 0$, in the region $R\ll r\ll 1/k$ of interest to us, the solution of the scattering problem can be written

$$g(z)\propto J_{|n|}(z)+4\nu(m-1)\varkappa^{2\nu}\mathcal{G}_{|n|,\nu}(z)+\sigma N_{|n|}(z).\tag{22}$$

Here we did not include the corresponding corrections to the Neumann function, since we can easily show that they contain higher orders of $\varkappa$ and are unimportant.

By comparing the approximate solution (18) valid for $0<x\ll 1/\varkappa$ with the solution (22) valid for $1\ll x\ll 1/\varkappa$ we can now find the scattering amplitude $\sigma_m(\varkappa)$. In analyzing this problem it is convenient to examine the different ranges of variation of the parameters separately.

1. The case $|n|<\nu$ incorporates both local modes with their numbers m taken from the interval $-\nu+1<m<1$ $(0<n<\nu)$ and nonlocal modes for which $-2\nu+1<m<-\nu+1$ $(-\nu<n<0)$. In finding the asymptotic solution (18a) of the equation we realize that far from the soliton the zero-frequency modes have the form

$$f_m^{(0)}\approx\frac{2}{x^{n+1}}\left(1-\frac{1}{x^{2\nu}}\right).\tag{23}$$

Hence we arrive at an approximation for $\Phi^{(0)}(x)$ in the important region $x\gg 1$:

$$\Phi^{(0)}(x)\approx\Phi_0-\frac{2}{n}x^{-2n}+\frac{4}{\nu+n}x^{-2n-2\nu},$$

$$\Phi_0=\frac{2\pi(\nu-n)}{\nu^2\sin(\pi n/\nu)},\tag{24}$$

where the contribution of the constant Φ_0 is crucial when $n>0$ but is a small correction when $n<0$.

Integration in (18a) with the use of (23) and (24) leads to an expression for $g_m^{\varkappa}$:

$$g_m^{\varkappa}(x)\propto\Phi_0x^n-\frac{2}{n}x^{-n}.\tag{25}$$

Here we have ignored terms of the form $x^{-2\nu}$ in comparison to $x^{-2|n|}$.

Let us compare the asymptotic solution we have just found with the solution (22) of the scattering problem. Using the expansion of cylinder functions for small values of z and comparing the resulting asymptotic expressions, we conclude that the term $x^{|n|}$ in (25) is related to the Bessel function $J_{|n|}$, while the term $x^{-|n|}$ is related to the Neumann function $N_{|n|}$ and determines the scattering amplitude σ. A simple comparison leads to an asymptotic expression for the scattering amplitude σ:

$$\sigma_m^{\nu}(\varkappa)=\frac{\pi(n\Phi_0/2)^{n/|n|}}{|n|!(|n|-1)!}\left(\frac{\varkappa}{2}\right)^{2|n|},\tag{26}$$

with $-2\nu+1<m<1$ and $m\neq-\nu+1$. Note that allowance for the corrections $\mathcal{G}_{|n|,\nu}$ leads to a contribution to σ of order $\varkappa^{2\nu}$, which is insignificant in the given range of parameters.

2. The case $|n|>\nu$ is realized for $m<-2\nu+1$ and $m>1$. Integration in (18a) with the use of the same approximations (23) and (24) leads to the asymptotic solution

$$g_m^{\varkappa}(x)\propto x^{|n|}\left(1+\frac{\nu+|n|}{\nu+n}x^{-2\nu}\right),\tag{27}$$

in which only the leading corrections in $1/x$ are retained.

The asymptotic expression (27) is valid for $m\leqslant\nu$, when the zeroth solutions f_m^0 are described by the functions $f_m^{(0)}$. A similar calculation can be done for $m>\nu$, where for the zeroth functions we use $f_m^{(1)}$:

$$g_m^{\varkappa}=\frac{1}{xf_m^{(0)}}(1-\varkappa^2\Phi^{(1)}(x)),$$

$$\Phi^{(1)}(x)=\int_0^x f_m^{(0)}(\xi)f_m^{(1)}(\xi)\xi d\xi\quad\text{for}\quad m>\nu.\tag{28}$$

Calculating the integral in (28), we arrive at the asymptotic expression (27) for $\nu>1$. For $\nu=1$ the asymptotic solution for modes with $m>1$ is

$$g_m^{\varkappa}(x)\propto x^m\left(1+\frac{1}{x^2}-\frac{\varkappa^2}{m(m+1)}\ln x\right).\tag{29}$$

Thus, the asymptotic solutions (27) and (29) obtained for $|n|>\nu$ differ dramatically from the earlier solution (25): the solutions (27) and (29) do not contain terms of the form $1/x^{|n|}$ and hence cannot yield an asymptotic expression of the form $J_{|n|}+\sigma N_{|n|}$. This is possible only if in the solution (22) the correction $\mathcal{G}_{|n|,\nu}$ is balanced by the scattering term σN_n. Note that this is an extremely stringent condition: not only must the terms $1/x^{|n|}$ be balanced but also all terms of the form $x^{2k}/x^{|n|}$, where $0 \leqslant k \leqslant |n|-1$. Allowing for the term in $\mathscr{G}_{|n|,\nu}$ related to N_n [see (21)], we get

$$4\nu(m-1)\varkappa^{2\nu}\frac{\pi N_n}{8n(n^2-1)}\prod_{k=2}^{\nu}A_k+\sigma N_n=0,$$

which yields a formula for the scattering amplitude:

$$\sigma_m^\nu(\varkappa)=\mathscr{A}_m^\nu\left(\frac{\varkappa}{2}\right)^{2\nu}, \quad -2\nu+1<m<1, \quad m\neq-\nu+1,$$

$$\mathscr{A}_m^\nu=-\frac{\pi 2^\nu(2\nu-1)!!}{(\nu-1)!|m|(m+1)\dots(m+2\nu-1)}. \tag{30}$$

3. The special cases $|n|=\nu$ and $|n|=0$ where the solutions (30) and (26) become invalid include the translational mode ($m=1$), the local mode with $m=-\nu+1$, and the nonlocal mode with $m=-2\nu+1$. For the translational mode the exact solution (15) yields $\sigma=0$. A calculation done on the basis of (18a) and a comparison of the results with the solution (22) of the scattering problem lead in the other two cases to the following asymptotic expressions for σ:

$$\sigma_m^\nu(\varkappa)=\frac{\pi}{2\ln(1/\varkappa)}, \quad m=-\nu+1, \tag{31}$$

$$\sigma_m^\nu(\varkappa)=\frac{4\pi}{[(\nu-1)!]^2}\left(\frac{\varkappa}{2}\right)^{2\nu}\ln\frac{1}{\varkappa}, \quad m=-2\nu+1. \tag{32}$$

The above analysis of scattering in the long-wavelength limit makes it possible to calculate the scattering amplitude in the long-wavelength approximation, i.e., for $k\ll 1/R$. At this point in our discussion, several general remarks concerning the nature of soliton–magnon scattering are in order.

It was found that as $k\to 0$ the scattering amplitude $\sigma_m(k)$ tends to zero for all values of m and ν. In most cases the amplitude $\sigma_m(\varkappa)$ given by Eqs. (30) and (26) is a regular function of $\varkappa$. In contrast to (30) and (26), for parameter values specified by (31) and (32) there exists a derivative $d^p\sigma/d\varkappa^p$ that has a singularity. The order is $p=1$ for $m=-\nu+1$, with the scattering being at its maximum. Such nonanalytic behavior of $\sigma(k)$ was detected in the numerical analysis of scattering of magnons with $m=0$ by a vortex in an antiferromagnet with an easy-magnetization plane done in Ref. 18 (see also Ref. 32). The scattering intensity (in contrast to the case of magnetic vortices discussed in Refs. 18 and 19) is not at its maximum for partial waves with smallest values of m ($m=\pm 1,0$).

The very fact that for a partial wave with a given m the limit point $k=0$ serves as the local zero-frequency mode is not critical for the scattering intensity. In particular, the mode with $m=1$ (the well-known translational mode) does not undergo scattering.

We also note that for the case of scattering by a BP soliton there are no simple relationships that link the scattering intensities for $m=+|m|$ and $m=-|m|$. For scattering of magnons by a vortex in magnetic materials with easy-magnetization planes, such relationships were established by numerical analysis: for antiferromagnets $\sigma_m^\nu(k)=\sigma_{-m}^\nu(k)$ (Ref. 18), while for ferromagnets $\sigma_m^\nu(k)$ and $\sigma_{-m}^\nu(k)$ can be obtained from each other by changing the sign of the magnon frequency (Ref. 19)

In conclusion of this section, we give the general solution of the problem of the scattering of a plane spin wave by a BP soliton. It is convenient to formulate the solution in terms of the variable $\tilde\Psi=\Psi\exp\{i\nu\chi\}$, which becomes $(n_x+in_y)\exp\{-i\omega t\}$ as $r\to\infty$ and describes a spin wave propagating against the background of the homogeneous state with $\mathbf{n}\|\mathbf{e}_z$. The need to pass from Ψ to $\tilde\Psi$ can be explained by the fact that although far from the soliton the magnetization is homogeneous, $\mathbf{e}_3\|\mathbf{e}_z$, the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ depend on χ. With allowance for (9) and (17), the asymptotic solution for $r\gg R$ can be written

$$\tilde\Psi=\sum_{m=-\infty}^{\infty}C_m(J_n(kr)+\sigma_m^\nu(\varkappa)N_n(kr))\exp\{in\chi-i\omega t\}, \tag{33}$$

where $n=\nu+m$, and the C_m are arbitrary constants. Using asymptotic expressions for the cylinder functions in the region $r\gg 1/k$ and selecting C_n on the basis of a comparison of (33) with the asymptotic expression (16) for free motion, we can write the general solution of the problem of scattering of a plane spin wave:

$$\tilde\Psi=\left[\exp\{i\mathbf{k}\cdot\mathbf{r}\}+\mathscr{F}(\chi)\frac{\exp\{ikr\}}{\sqrt{r}}\right]\exp\{-i\omega t\},$$

$$\mathscr{F}(\chi)=\frac{\exp\{-i\pi/4\}}{\sqrt{2\pi k}}\sum_{m=-\infty}^{\infty}(\exp\{2i\delta_m^\nu\}-1)\times\exp\{i(\nu+m)\chi\}. \tag{34}$$

In (34) we have introduced the scattering phase $\delta_m^\nu(\varkappa)$, which is related to the scattering amplitude by the simple formula $\sigma=-\tan\delta$.

The total scattering cross section is given by the formula

$$\varrho=\int_0^{2\pi}|\mathscr{F}|^2d\chi=\sum_{m=-\infty}^{\infty}\varrho_m,$$

where the $\varrho_m=(4/k)\sin^2\delta_m^\nu$ are the partial scattering cross sections.

As noted earlier, for small k, the maximum scattering is related to the local mode with $m=-\nu+1$, for which, according to (31), the scattering phase $\sigma=\pi/2\ln\varkappa$. Hence, in the leading approximation in k it is enough to limit oneself to the contribution of this mode, with the result that we arrive at an expression for the scattering function of the form

$$\mathscr{F}(\chi)\approx\sqrt{\frac{\pi}{2k}}\frac{\exp\{i(\chi+\pi/4)\}}{\ln kR}, \quad k\ll 1/R. \tag{35}$$

In this approximation the scattering is isotropic ($|\mathscr{F}(\chi)|$ is independent of χ). The corrections to this expression are of order $1/(kR)^{2\nu+1/2}$ and are important only for determining the anisotropy of $\mathscr{F}(\chi)$.

The total scattering cross section (which has an integrable singularity) in the limit $\varkappa\to 0$ is given by the formula

$$\varrho(\varkappa)\approx\frac{\pi^2}{k\ln^2 kR},\quad k\ll 1/R. \tag{36}$$

5. ANALYSIS OF SCATTERING DATA FOR MODERATE VALUES OF k

The scattering can also be treated analytically in the short-wavelength limit, $k\gg|m|/R$. It is natural to assume that in this case the problem can be analyzed in the quasiclassical approximation, which yields

$$g_m^\varkappa\propto\sqrt{\frac{p(x)}{x}}\cos\left[\text{const}+\int_{x_0}^{x}p(\xi)d\xi\right],$$

$$p^2(x)=\varkappa^2-\mathscr{U}_m(x)+\frac{1}{4x^2}. \tag{37}$$

Indeed, analysis shows that (37) is valid for all values of $z=kr$ larger than the coordinate of the turning point, $z_0=\varkappa x_0$, which corresponds to the condition $p(x_0)=0$. The value of x_0 is small, $x_0\sim|m|/\varkappa\ll 1$.

On the other hand, at small distances $r\ll R$ ($x\ll 1$) the ''potential'' $\mathscr{U}_m$ has the asymptotic form $\mathscr{U}_m\approx(\nu-m+1)^2/x^2$, i.e., it describes free magnons of the form (16) with a mixed index:

$$g_m\propto J_{|\nu-m+1|}(z),\quad f_m\propto J_{|\nu-m|}(z)\quad \text{for } r\ll R.$$

For $k\gg|m|/R$, there is a broad range of values of r, $|m|/k\ll r\ll R$, in which we can limit ourselves to the asymptotic expression for the Bessel function in the limit $z\gg 1$ and $z\gg|m|$:

$$g_m^\varkappa\propto J_{|\nu-m+1|}\approx\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{1}{2}|\nu-m+1|-\frac{\pi}{4}+\frac{4(\nu-m+1)^2-1}{8z}\right). \tag{38}$$

To within terms of order $1/z^2$, the solutions (37) and (38) coincide in the entire range of parameter overlap. Hence, doing the asymptotic expansion of (37) far from the soliton, we arrive at the short-wavelength asymptotic expression for the scattering amplitude:

$$\sigma_m^\nu(\varkappa)\approx\frac{\pi(m-1)}{\sin(\pi/2\nu)}\frac{1}{\varkappa},\quad \varkappa\gg|m|. \tag{39}$$

Most importantly, this formula reproduces a property of the exact solution (15) according to which $\sigma_m^\nu=0$ holds at $m=1$. More than that, the scattering amplitude asymptotically tends to zero as $1/\varkappa$ for all $m\neq 1$, with the σ being equal in absolute value but having opposite signs for magnon modes with $m=|m|$ and $m=-|m|+2$. Below we will see that this result plays an important role in the analysis of density of magnon states in a 2D magnetic material.

Now we can compare the scattering amplitudes in the long- and short-wavelength limits. Clearly, $\sigma\to 0$ in both cases, but the signs of $\sigma(\varkappa)$ for $\varkappa\to 0$ and $\varkappa\to\infty$ are opposite. This situation is characteristic of magnon scattering by a 1D soliton in the sine-Gordon and ϕ^4 models and of the Landau–Lifshitz equation (see the review article in Ref. 3). It can be assumed that for a certain finite $k=k_p$ the scattering amplitude has a pole. Naturally, there is no real divergence at this pole: the physically observed scattering phase δ_m^ν varies monotonically. The existence of a pole means that the total increment of the scattering phase, $\delta(\infty)-\delta(0)$, is finite. According to numerical calculations for a soliton with a topological charge $\nu=1$, this increment is equal to π (to within sign), i.e., each mode is associated with a single pole. Such a singularity manifests itself in the analysis of the number of magnon degrees of freedom (see Sec. 7).

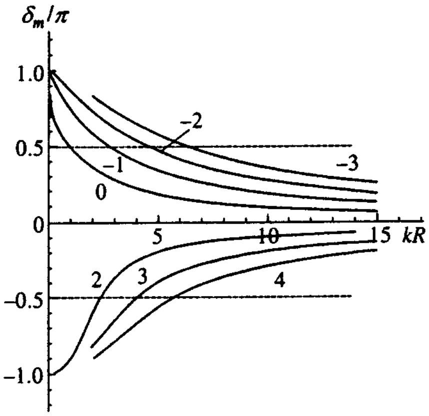

FIG. 1. Plots of δ_m vs. kR for $\nu=1$, labeled with the corresponding values of m. The dashed straight lines drawn through the values $|\delta_m|=\pi/2$ designate the positions of the poles of the scattering amplitude.

To analyze the intermediate values $kR\sim 1$, we solved the scattering problem numerically. The calculations were done by numerical integration of the spectral equations for the initial problem [Eq. (10)] and the modified problem [Eq. (13)] within a broad range of values of kR and m: $10^{-3}<kR<10^3$ and $-20\leq m\leq 20$ (the results of each calculation agree with what was said earlier). Basically we are interested in case with $\nu=1$, where the soliton energy is at its minimum. However, some data were obtained for $\nu=2$, 3, and 4, too.

Numerical calculations verified the long- and short-wavelength asymptotic expressions for the scattering amplitude given above. In the intermediate region of wave-vector values, $k\sim 1/R$, there are poles in the scattering amplitudes at $k=k+p$ for all the modes in question (Fig. 1 depicts the data for the modes with different values of m in the case of a soliton with $\nu=1$).

Let us discuss the problem of the position of the poles in the scattering amplitude in greater detail. According to the numerical data at $\nu=1$, for all $m\neq 1$ there is only one pole at $k=k_p$. Here k_p increases with $|m|$, and the functions $k_p=k_p(m)$ are different for $m=+|m|$ and for $m=-|m|$ (the reader will recall we are dealing with solitons with $|\nu|=1$). For very large values of $|m|$ the pole goes to infinity (Fig. 2). The situation becomes more complicated when $\nu>1$. More precisely, preliminary numerical data show that for a given m there can be several poles, with their number N_m not exceeding ν.

For comparative analysis of the scattering of modes with different values of m, we write explicitly the asymptotic expression for the scattering phase at $\nu=1$:

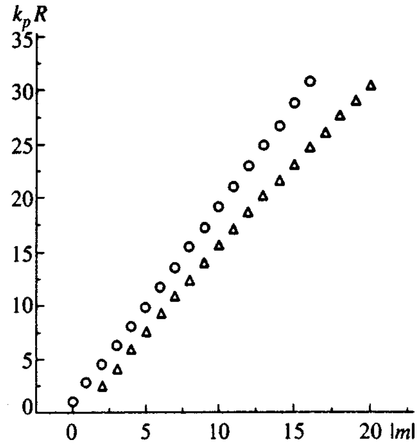

FIG. 2. The positions of the poles k_p as a function of the mode number m at $\nu=1$. The △ correspond to $m>0$ and the ○ to $m\leqslant 0$.

$$\delta(\varkappa)\approx\begin{cases}\pi\,\mathrm{sgn}\,m\left(1-\dfrac{\varkappa^2}{2m(m+1)}\right), & \varkappa\ll 1, \qquad m\neq -1,0,\\ \dfrac{\pi(1-m)}{\varkappa}, & \varkappa\gg|m|.\end{cases}\tag{40}$$

Assuming that these equations are valid at least qualitatively and setting $\varkappa\sim 1$, we can make a rough estimate of the position of the pole by equating the values of $\delta(\varkappa)$ for $\varkappa\ll 1$ and for $\varkappa\gg 1$. This yields $k_p\approx|m|/R$ for $|m|\gg 1$. Such an estimate reproduces fairly accurately the linear increase in k_p as a function of $|m|$ for large values of $|m|$ (see Fig. 2).

6. MAGNON MODES IN A MAGNETIC MATERIAL OF FINITE SIZE

The foregoing analysis of the scattering problem in the long-wavelength limit can be used to study the natural magnon modes in a magnetic material of a finite surface area containing a soliton. Such a problem plays an important role in many applications. Firstly, its solution can be used to describe analytically the data obtained through computer simulations of soliton motion, which are always done for systems of finite dimensions. In particular, in Refs. 33 and 19, this approach was used to describe the dynamics of a vortex in a ferromagnet with an easy-magnetization plane and to verify the non-Newtonian equations of motion containing third derivatives of the vortex coordinates with respect to time. Secondly, as noted earlier, this calculation can be used directly to describe the natural modes for the small particles of the magnetic material, which are in what is called the vortex state.[34]

We begin with the simplest case of the magnon modes in a circular system with a finite radius L and a soliton at the center. We discuss both the Dirichlet boundary conditions

$$\Psi(r,\chi)|_{r=L}=0,\tag{41}$$

which correspond to a fixed value of magnetization at the boundary, and the Neumann boundary conditions

$$\left.\frac{\partial\Psi(r,\chi)}{\partial r}\right|_{r=L}=0,\tag{42}$$

which model the case of free boundary conditions. There is no difficulty in extending these results to the case of general boundary conditions, but we do not do this here. The magnon spectrum in such a system is discrete. In the absence of solitons, the characteristic wave numbers $k_{m,i}$ are equal to $j_{m,i}/L$, where $j_{m,i}$ is the ith zero of the Bessel function J_m or the derivative of this function for the case (41) or (42), respectively.

In a magnetic material with a BP soliton, when k is large, we can ignore the local part of the function and write $J_n(kL)+\sigma(k)N_n(kL)=0$. It is natural, then, to expect the same behavior from k, i.e., $k=j/L$, where j lies between the values of the corresponding root of the Bessel or Neumann functions or the derivatives of these functions.

However, for $-\nu<m\leqslant\nu$, i.e., in the case of zero-frequency modes, the symmetry of the problem is high (scale invariance is restored). Hence we should expect the occurrence of Goldstone modes. In an unlimited (infinite) magnetic material, the frequencies of the Goldstone modes are zero, while in the presence of a boundary these modes manifest themselves as modes with very low frequencies, i.e., $kL\ll 1$. In particular, such modes arise for a vortex in a ferromagnet with an easy-magnetization plane in the case where $|m|=1$, which corresponds to translational motion of the vortex. For this mode, $k\sim r_v/L^2\ll 1/L$, where r_v is the radius of the vortex core. Since in this case the solution is approximated by (17) with an exponential accuracy even for $r>r_v$, the existence of Goldstone modes is determined solely by the scattering matrix.

When we are dealing with a BP soliton, in the analysis of Goldstone modes it is not enough to limit oneself to a solution in the form (17) corresponding to the scattering problem—one must also allow for the local part of the solution. The corresponding calculations are so tedious that in studying Goldstone modes it is more convenient to deal with the long-wavelength asymptotic expressions derived earlier; the expressions are valid for $r\ll 1/k$, i.e., for $kr<kL\ll 1$. It is this region that is so important in the analysis of such modes. Note that no Goldstone modes are present in the modified problem (the long-wavelength asymptotic expression (18a) has no small parameter, with the result that the boundary condition $g_m^{\varkappa}=0$ leads only to the solution $k\sim 1/L$).

In analyzing the Goldstone modes it is convenient to return to the initial problem for the function $f_m^{\varkappa}$. In this range of values of k, it is only natural to use the approximate expression (18b) for $f_m^{\varkappa}$. The analysis done using this expression shows that Goldstone modes occur only in the region where local modes exist. In the case of the Dirichlet boundary conditions, the spectrum of the Goldstone modes, which can be found from the condition $f_m(kL)=0$, has the form

$$kL=\begin{cases} 2\nu\sqrt{\frac{1+n}{\nu-n}\,\frac{\sin(\pi n/\nu)}{\pi}}\left(\frac{R}{L}\right)^n, & -\nu+1\leqslant m\leqslant\nu,\\ \sqrt{\frac{2}{\ln(L/R)}}, & m=-\nu+1.\end{cases} \tag{43}$$

The situation is somewhat more complicated for free boundary conditions. In particular, with Neumann boundary conditions, the solution (18b) does not allow for states with $kL\ll 1$. In this case, however, we can derive a solution by using a cylinder function of imaginary argument, which yields $\omega=Dk^2<0$ for the case of a ferromagnet or $\omega^2<0$ for an antiferromagnet. Below we discuss the physical meaning of negative values of ω and ω^2.

The following roots of the equation already agree with the condition $kL\sim 1$. They correspond to $k^2>0$ for all types of boundary conditions. Since for $k\sim 1/L$ and $R\ll L$ the ratio r/R is large at the boundary, the value of k_pL is close to the value of the corresponding zero of the Bessel function, $j_p\equiv j_{\nu+1,p}$, where $J_{\nu+1}(j_p)=0$, or to the value of the zero of the derivative, j'_p, where $J_{\nu+1}(j'_p)=0$ in the case of fixed and free boundary conditions, respectively:

$$k_pL=j_p+\frac{2\nu}{kL}\,\frac{J_\nu(j_p)}{J'_{\nu+1}(j_p)}\left(\frac{R}{L}\right)^{2\nu}, \tag{44a}$$

$$k_pL=J'_p+\frac{2\nu}{kL}\,\frac{(R/L)^{2\nu}}{J''_{\nu+1}(j'_p)}\left\{\frac{2\nu+1}{kL}J_\nu(j'_p)-J'_\nu(j'_p)\right\}. \tag{44b}$$

Thus, the spectrum of the natural frequencies of a small particle of a magnetic material in an inhomogeneous state contains anomalously low frequencies, which manifest themselves in the magnetic resonance of samples containing such particles, say, ferroliquids and granular magnetic materials. Usov and Peschany[34] found that the magnetization distribution in a particle in the vortex state is well approximated by the BP soliton. Although our calculations can be applied only to particles shaped as a thin disk, it is easy to generalize them to the case of a cylinder.

Now we go back to the discussion of the meaning of the result $k^2<0$ for a Goldstone mode for free boundary conditions. We examine the most interesting case, $m=1$, corresponding to the translational motion of a BP soliton (below we will show that the parameters of a Goldstone mode can be directly related to the equations of motion of the soliton). The parameters of what is known as the translational Goldstone mode can be obtained directly from the exact solution (15). For $kR\ll 1$ the solution is

$$\Psi(x)\propto r^{\nu+1}\left(1-\frac{4\nu(\nu+1)}{(kr)^2}\,\frac{1}{(r/R)^{2\nu}+1}\right),$$

which implies that for fixed boundary conditions,

$$k^2=\frac{4\nu(\nu+1)}{L^2}\left(\frac{R}{L}\right)^{2\nu}. \tag{45}$$

For free boundary conditions the solution has the same form but k^2 is negative. Negative k^2 is not inconsistent with the presence of Bessel functions of imaginary argument (modified Bessel functions) in the solution, since we are studying this solution in the region $|k|\lesssim 1/L$, where the exponential increase of the function $I_n(z)\propto\exp\{z\}$ for $z\gg 1$ does not manifest itself.

For ferromagnets and antiferromagnets these results lead to significantly different physical pictures of soliton dynamics, which means that cases must be analyzed separately.

In the case of an antiferromagnet, there are two frequencies corresponding to the translational Goldstone mode:

$$\omega_0^2=\pm\frac{4\nu(\nu+1)c^2}{L^2}\left(\frac{R}{L}\right)^{2\nu}. \tag{46}$$

Clearly, this frequency has meaning only for fixed boundary conditions, and negative ω^2 mean that the system is unstable. At the same time, for a ferromagnet the value $\omega=Dk^2<0$ does not contradict the condition for stability. These results can easily be explained on the basis of a simple physical picture of soliton motion.

Obviously, for an antiferromagnet, which is described by Lorentz-invariant equations, the dynamics of all excitations must also be Lorentz-invariant. When the soliton is slow, $v\ll c$, this means that in the leading approximation the soliton coordinate $\mathbf{X}$ (for $\mathbf{X}$ the origin is at the center of the system) in the case of an antiferromagnet satisfies an equation of the Newtonian type:

$$M\frac{\partial^2\mathbf{X}}{\partial t^2}=\mathbf{F}_e, \tag{47a}$$

where $\mathbf{F}_e$ is the external force acting on the soliton, and $M=E_0/c^2$ is the effective soliton mass, with E_0 the soliton energy [see Eq. (3)]. Assuming that when the deviation of the soliton from the equilibrium position at the center of the system is small we can write

$$\mathbf{F}_e=\frac{\alpha\mathbf{X}}{L^p}, \tag{47b}$$

let us compare the value of the frequency obtained by (46) with the value of $\omega^2=-\alpha/ML^p$. We find that $p=2(\nu+1)$, and $\alpha=\mp 16\pi\nu^2(\nu+1)AR^{2\nu}$ for the Dirichlet and Neumann boundary conditions, respectively. This corresponds to the simple picture according to which $\mathbf{F}_e$ is the force of the image acting on the soliton because of the presence of a boundary. Since magnetic vortices interact as 2D charges and a BP soliton with $\nu=1$ is a vortex dipole, solitons with given $\nu>1$ can be interpreted as 2ν-multipoles, which explains the presence of p in (47b) and the sign of α.

Thus, the properties of the translational Goldstone mode in an antiferromagnet can easily be understood from the following reasoning. When a soliton is deflected from its equilibrium position $\mathbf{X}=0$, it is driven by the force of the image. For the Dirichlet boundary conditions the force is a restoring one (repulsion from the boundary) and the motion is stable. If the soliton is attracted to the boundary (the Neumann boundary conditions), Eq. (47) describes the departure of the soliton from the unstable position of equilibrium at $\mathbf{X}=0$.

Allowance for the next values of $k_{n,i}$ for $i>0$ can also be explained on the basis of effective equations for $\mathbf{X}$. Here the hierarchy of the effective equations of motion containing only even-order time derivatives manifests itself. The coefficients of the higher-order derivatives diverge as $L\to\infty$. Mertens *et al.*[33] proposed equations of this type for describing the behavior of interplanar vortices in a ferromagnet.

The situation is quite different for a ferromagnet. The equation that is commonly used to describe the soliton dynamics is

$$M\frac{\partial^2\mathbf{X}}{\partial t^2}+G\left(\mathbf{e}_z\times\frac{\partial\mathbf{X}}{\partial t}\right)=\mathbf{F}_e\,. \tag{48}$$

Here $\mathbf{F}_e$ is the external force, which, obviously, is the same as in the case of an antiferromagnet [see Eq. (47b)], and G is the gyroscopic term, whose value is determined only by the topology and has been reliably established,[3,24,35,36] $G=4\pi\nu A/D$. The data on the effective mass of 2D solitons and vortices are contradictory: in Ref. 37 it is stated that in a ferromagnet with an easy-magnetization plane the value of M is finite but diverges as the anisotropy constant K tends to zero, $M\propto 1/K$. In Ref. 38 the result for a vertex is $M\propto 1/L$, in Ref. 33 the mass M is proportional to $\ln L$, and in Ref. 19 M is finite, but only if the term $G_3(\mathbf{e}_z\times\partial^3\mathbf{X}/\partial t^3)$ is present in the effective equations of the form (48). In Refs. 36 and 39, the dynamics of a BP soliton is described on the basis of the Hamiltonian formalism with noncanonical Poisson brackets, and the relationship between momentum and velocity and the values of the mass are not discussed.

According to (45), the frequency of the translational Goldstone mode for a ferromagnet has the form

$$\omega_0=\pm\frac{4\nu(\nu+1)D}{L^2}\left(\frac{R}{L}\right)^{2\nu}, \tag{49}$$

with the ''plus'' and ''minus'' corresponding to the Dirichlet and Neumann boundary conditions, respectively. In the present case there is no instability, since Eq. (48) with $M=0$ (i.e., only the gyroscopic term is taken into account) describes small oscillations of the soliton in the case of attraction to the boundary and in the case of repulsion from the boundary. Allowance for the next translation mode, whose frequency is determined by the formula

$$\omega_1=D(j/L)^2 \quad \text{or} \quad \omega_1=-D(j'/L)^2 \tag{50}$$

in the case of the Dirichlet or Neumann boundary conditions, respectively, makes it possible to draw a conclusion about the inertial terms in the equation of motion.

Assuming that $\omega_0\ll\omega_1$, these roots can easily be compared with the two frequencies that arise in the solution of Eq. (48). Indeed, in this case we have $\omega_0\approx-\alpha/GL^p$, which yields exactly the first value of the frequency of the translational Goldstone mode. For the second value we get $\omega_1\approx-G/M$. This value can be compared to (50) if we put

$$M=-\frac{4\pi\nu A}{D^2}\left(\frac{L}{j}\right)^2 \quad \text{or} \quad M=\frac{4\pi\nu A}{D^2}\left(\frac{L}{j'}\right)^2, \tag{51}$$

respectively, for fixed or free boundary conditions. Thus, as for a vortex, the dynamics with the frequency ω_1 is determined by the entire region to which the magnetic material is confined. Just as the coefficient G_3 in the third-order equations for vortices in a ferromagnet is nonlocal, so is the coefficient M: it depends on the boundary conditions and diverges as $L\to\infty$. The divergence of M is probably a general property of 2D magnetic materials with a gapless dispersion law.

We also note that the finite value of the soliton mass $M\propto 1/K$, where K is the anisotropy constant, obtained in Ref. 37 for a magnetic material with an easy-magnetization, axis does not contradict the above dependence $M\propto L^2$ for an isotropic ferromagnet. Indeed, in a magnetic material with an easy-magnetization axis, the gap in the magnon spectrum is finite and a characteristic linear scale $\Delta_0=\sqrt{A/K}$ appears, from which we can obtain the same result as in Ref. 37, $M\propto\Delta_0^2\propto 1/K$, if L is replaced by Δ_0 in (51).

7. DENSITY OF MAGNON STATES OF A 2D ISOTROPIC MAGNETIC MATERIAL IN THE PRESENCE OF A SOLITON

A 2D magnetic material can be described thermodynamically with allowance for soliton excitations via a generalization of soliton phenomenology developed by Krumhansl and Schrieffer[1] and Currie *et al.*[2] for 1D systems to the two-dimensional case. According to their approach, at low temperatures the state of a 1D magnetic material can be described in terms of almost free excitations, magnons and kinks. The main effect of their interactions manifests itself in the form of an asymptotic shift of the phase of a magnon scattered by a kink. This causes the total number of magnon states from the continuous spectrum to change (in comparison to the case of a magnetic material without a soliton) by $\Delta N=\int_{-k_0}^{k_0}\rho(k)dk$, where $\rho(k)=(1/2\pi)d\delta(k)/dk$ is the density of states. This quantity is a negative integer, i.e., the number of magnon states in the presence of a soliton decreases by ΔN, which is obvious, since a fraction of the magnon states are now described as the collective modes of the kink dynamics. The variation of the density of magnon states due to the addition of a kink to the system causes a change in the thermodynamic characteristics of the magnon gas, in particular, the free energy of the magnons. In the phenomenological approach, this change in the free energy of magnons is interpreted as a change in the kink energy due to kink–magnon interaction.

Let us use all these ideas in the 2D case. Clearly, in a 2D magnetic material the total number of states is proportional to L_xL_y. A free magnon corresponds to the expansion (16) in the cylinder harmonics $J_m(kr)\exp\{im\chi\}$ in which the angular variable has already been quantized, so that only the radial part $J(kr)$ needs to be quantized. In a circular geometry with radius L, the simplest quantization condition (41) has the form $J_m(kL)=0$, from which it follows that $k_nL=j_{m,n}$. In the region of interest to us, $n\gg 1$, the zeros of the Bessel functions, $j_{m,n}$, are approximately equal to πn. From this fact we can formally determine the admissible values of the wave number by the same expression as in the 1D case. However, one must bear in mind that such an approximation for $j_{m,n}$ is valid only when m is not very large. For modes with $|m|\gg 1$ the first zero $j_{m,1}\approx|m|$. Hence in a system whose size L is finite there is a restriction on the admissible numbers of the modes, namely, $|m|\leqslant L$. Allowing for this fact, we arrive at a rule for summing over the magnon states for a 2D magnetic material without a soliton:

$$\sum_{k,m} = \frac{L}{\pi}\int_0^{k_0} dk \sum_{m=-kL}^{kL} .$$

Naturally, for the total number of magnon states we arrive at the usual formula $N_{2D}=L^2k_0^2/\pi$.

Allowance for the soliton–magnon interaction leads to a shift in the magnon phase and changes, just at its does in 1D systems, the expression for the density of states (in our case, partial states for magnons with a given m) $\rho_m(k) = (1/\pi) d\delta_m(k)/dk$. The total density of magnon states is found by summing over m:

$$\mathscr{R}(k)=\sum_{m=-kL}^{kL} \rho_m(k)=\frac{1}{\pi}\sum_{m=-kL}^{kL}\frac{d\delta_m(k)}{dk}. \qquad (52)$$

Note that the density of states $\mathscr{R}(k)$ in the long-wavelength region has an (integrable) divergence caused by the mode with $m=0$, for which, according to (31), $\rho_0(k) \approx (2k)^{-1}\ln^{-2}(kR)$ diverges in the limit $kR\rightarrow 0$ [cf. (36)]. It is also obvious that at low temperatures, $T \ll T_*$, where $T_* = \hbar D/R^2$ for ferromagnets and $T_* = \hbar c/R$ for antiferromagnets, it is enough to limit oneself to the long-wavelength approximation. In particular, in the adopted approximation, the density of energy states can be written

$$g(E) \propto \frac{1}{ER\ln^2(E/T_*)}. \qquad (53)$$

In principle, the density of states for an arbitrary k can be calculated numerically. Here the presence of a pole in the amplitude of scattering of magnons with a given m means that the total phase changes by $+\pi$ or by $-\pi$ as k changes from zero to infinity, with the modes with $m>1$ and $m<1$ providing contributions to $\mathscr{R}(k)$ that are opposite in sign. Thus, for values of k that are not small the total number of magnon states does not decrease (as it does in the 1D case); rather, the magnon modes are redistributed among the states with different values of m. In general the signs in the series (52) are found to alternate. In thermodynamic calculations the temperature acts as a sort of regularizing factor in this summation process. The main contribution of the various modes, in particular, the change of the number of partial states by one unit, manifests itself in the order in which the poles k_p appear in the scattering amplitude as k increases. Since k_p increases with m (see Sec. 5), the contributions of the modes with an ever increasing m manifest themselves successively as the temperature rises.

8. CONCLUSION

Thus, we have constructed the soliton–magnon scattering matrix for the simplest but physically interesting 2D model of an isotropic magnetic material. The analysis has been carried out both for the Landau–Lifshitz equation, used to describe ferromagnets, and for the Lorentz-invariant σ-model, used in field theory and to describe antiferromagnets. We are the first to obtain an exact solution of the scattering problem for the partial mode with the azimuthal quantum number $m=1$. Note that such solutions are not known for all one-dimensional problems.

What is important is that the possibility of such an investigation is not related to exact integrability of the problem. Indeed, the model of an isotropic magnetic material is exactly integrable in the static case, $\mathbf{n}=\mathbf{n}(x,y)$, but nothing is known of its integrability in the case $\mathbf{n}=\mathbf{n}(x,y,t)$.

We have calculated the scattering amplitude for $m\neq 1$ (analytically in the long-wavelength approximation $kR \ll 1$ and for large values of kR and also numerically for arbitrary values of kR). We have found that the partial scattering amplitudes have poles (the scattering phases pass through $\pi/2$) at certain values $k=k_p$, with k_p increasing with m approximately by a linear law. This is enough to calculate the magnon density of states in the presence of a soliton.

We have used our results to describe various physical properties of solitons and local magnon modes. In particular, we have calculated the frequencies of the magnon modes for a magnetic material of finite dimensions. What we have found is that in the small particles of ferromagnets containing a soliton (particles in what is known as the vortex state, whose properties are being widely discussed at present) natural modes arise with anomalously low frequencies. The data on the frequencies of the local modes have been used to derive the equations of motion of a soliton in a ferromagnet. We have calculated the magnon density of states in the presence of a soliton, which makes it possible to construct a soliton phenomenology for 2D magnetic materials that allows for the soliton–magnon interaction.

There are other possible applications of our results worth noting. In some of the papers (see, e.g., the review article in Ref. 7) devoted to the study of ordered 1D media including magnetic materials, several nonequilibrium characteristics of a soliton gas, primarily, the coefficients of diffusion and viscosity, were investigated. The theories developed by the researchers were based on using the exact wave functions of magnons against the background of a soliton. The asymptotic expressions for the wave function for small k derived in the present paper have made it possible to study the irreversible process for the 2D gas of elementary excitations, including solitons and magnons, in isotropic magnetic materials at low temperatures.

The results concerning the σ-model can easily be extended to the Euclidean case and can be used to describe the quantum properties of spin chains with antiferromagnetic interaction. The properties of such systems are determined by the instantons of the Euclidean version of the nonlocal σ-model. Also widely discussed are instantons with a structure of the BP soliton (see Ref. 40) and what is known as merons, which have a half-integer topological charge (see Ref. 41). To calculate the pre-exponential factors in the corresponding transition amplitudes (the fluctuation determinant), we must know the complete set of eigenstates against the instanton background. Most important are zero-frequency modes (for more details see Ref. 42). Hence our results, especially concerning the nontrivial local zero-frequency modes, may prove to be important in developing the instanton approach in the quantum theory of 1D magnetic materials.

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1)If this fact is not taken into account, the amplitude for scattering of the translational mode by a BP soliton turns out to be finite,[32,22] whereas according to (15) it must be zero.

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