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Two-dimensional Magnetic Solitons and Thermodynamics of Quasi-two-dimensional Magnets

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Abstract—The article reviews two-dimensional magnetic solitons in a classical weakly-anisotropic Heisenberg magnets. Topological classification, structure, dynamical properties and thermodynamical contribution of 2D solitons to response functions of the magnet are discussed. Based on effective equations of motion we calculated the soliton contribution to the dynamical structure factor of ferromagnets and antiferromagnets both for localized topological solitons and magnetic vortices.

1. INTRODUCTION

Different mathematical and physical aspects of the theory of solitons have been discussed by many investigators and are represented in a number of review articles and monographs. One of the principal applications of solitons is connected with the important role of solitonical solutions in the physics of quasi-one-dimensional (1D) ordered systems, first of all, 1D magnets, see reviews [1, 2]. The topological 1D solitons, so-called kinks, are important for the thermodynamics of 1D magnetism. Kink-type solutions are well known for a number of exactly integrable nonlinear 1D (in x, t-geometry) equations such as the sine–Gordon equation and the Landau–Lifshitz equation, and are widely used for the construction of the solitonical thermodynamics of 1D magnets.

In recent years considerably increased attention has been paid to quasi-two-dimensional (2D) magnets. It has been connected with (i) technical advances in the production of layered materials including intercalated graphite compound [3], artificial 2D spin systems such as magnetic lipids [4], (ii) important experimental information obtained by inelastic neutron-scattering, electron and nuclear magnetic resonance measurements and (iii) the existence of 2D spin subsystems in high-temperature superconductors. At the present time two-dimensional (in x, y, t-geometry), exactly integrable evolution equations, which admit localized solitonical states with finite energy and can be used for the real magnets are unknown. (Note that the familiar Ishimori equation [5] for the unit vector $\mathbf{n}$, $|\mathbf{n}|^2 = 1$ contains also some scalar field).

However, 2D Landau–Lifshitz and σ-model equations, which are widely used for antiferromagnets (AFM), have some homotopically nontrivial solutions, see refs [6–11]. Such solutions describe the finite energy states; they make an essential contribution to the physical properties of 2D magnets.

The solitonical approach to the thermodynamics of 2D magnets based on the knowledge of the soliton gas with finite density was taken apparently for the first time by Kosterlitz and Thouless [12]. However, it was actively used only during recent years [13–22].

At the present time the theory of solitons of 2D magnet is far from the completion.

Different magnets show various solitons of magnetic vortex type with essentially different properties. We will discuss both easy-axis and easy-plane magnets.

In a classical Heisenberg magnet with weak anisotropy the dynamical variable represents an unit vector **n** which is the normalized magnetization **m** in the case of an ferromagnet (FM) and normalized sublattice magnetization **l** in the case of the AFM.

The present article is devoted to the consideration of magnetic soliton dynamics for different types of magnets. On the basis of effective equations of motion we calculated the soliton contribution to the response functions of the magnet.

The general outline of the article is as follows: Classification of different types of 2D magnetic solitons from the point of view of homotopical topology is carried out in Section 2. Section 3 formulates the approach to the problem, basic model assumptions; specific models of magnetics are also represented. The structure of soliton-type solutions is discussed in Section 4. We will take into account dissipation processes in detail. Moreover, the simplest version of the soliton perturbation theory approach is formulated.

Investigation of soliton dynamics in two topologically different cases of localized topological solitons (LTS) and magnetic vortices (MV) is carried out in Section 5. We will describe soliton structure in special models; exact solutions will be displayed. Effective equations of motion, calculated in Section 5 differ strongly for localized solitons and vortices. The point is that the effective equation of vortex motion does not contain the inertial term. Hence, stationary stable motion is only possible. This problem is considered from the point of view of effects of vortex 'freezing into the condensate'. Soliton contribution to response functions of 2D magnet for LTS and MV is examined in Section 6. We can see again principal distinctions between two types of solitons. In-plane spin correlations are globally sensitive to the presence of vortices, but is locally sensitive to localized solitons.

2. CLASSIFICATION OF 2D SOLITONS

As in the 1D case there are both topological solitons and stable magnetic excitations without topological stability in 2D magnets. A number of results concerning the structure of 2D magnetic solitons are reviewed in refs [6 and 23]. Considering only the topologically stable solitons let us point out two types of such excitations.

A nonlocalized topological soliton or magnetic vortex shows nonuniform dynamical variable in the distance from the soliton. Such a soliton can exist in a magnet with continuous degeneracy of the ground state. One of the examples is the easy-plane magnet. Its ground state is degenerate with respect to angular variable φ, which determines the direction of the vector **n** in the easy plane. The topological properties of such solitons as the properties of vortices in a superfluid Bose-gas are characterized by homotopic group π_1 and determined by the distribution of φ on the contour γ, which covers the soliton and is drawn in the large distance from its centre. Since the degeneracy space of the easy-plane magnet is circumference S^1 that the homotopical group π_1 is isomorphic to the integer group. Such a soliton is characterized by integer topological charge ν, see reviews [6, 10, 11, 23]. Let us note that the analogy between easy-plane magnets and superfluid systems, discussed from different points of view in refs [24, 25] is of great importance to the physics of 2D nonlocalized solitons. There are solitons of MV type in pure easy-plane magnets. The basic plane anisotropy leads to the vortex–domain wall system binding.

In the case of the uniform dynamical variable distribution in the distance of the soliton such an excitation becomes unstable in the meaning of the homotopic group π_1. However, even under the condition $\mathbf{n}(\mathbf{r}) \to \mathbf{n}_0 = \text{const.}$ when $|\mathbf{r}| \to \infty$ the solution can exhibit

topological characteristics. The solution of such type will be called a localized topological soliton. Let us explain the topological nature of the LTS. Here we note that the condition $\mathbf{n}(r) \to n_0$ leads to the identification of all points of the 2D magnetic plane (x, y) at infinity. Since such a plane is equivalent to the sphere S^2 we must analyse mapping of this sphere S^2 to the sphere $|\mathbf{n}|^2 = 1$ in $\mathbf{n}$-space. This mapping is characterized by the homotopical group π_2, see [6, 11]. In specific models of FM and AFM the group π_2 is isomorphic to the integer group. Consequently LTS as the MV is described by integer topological charges.

3. THE MODEL

Let us analyse the classical weakly-anisotropic Heisenberg magnets. The dynamical description of such a system is carried out on the basis of the Lagrangian for the unit vector $\mathbf{n}$. It is convenient to use angular variables for $\mathbf{n}$, $n_z = \cos\theta$, $n_x + in_y = \sin\theta \times \exp(i\varphi)$, where the polar axis is directed along the selected axis of the crystal.

The dynamics of a wide class of uniaxial magnets can be described on the basis of the Lagrangian

$$\mathcal{L} = \alpha a M_0^2 \int d^2x \left\{ \frac{1}{2c^2}\left[\left(\frac{\partial\theta}{\partial t}\right)^2 + \sin^2\theta\left(\frac{\partial\varphi}{\partial t}\right)^2\right] - \tfrac{1}{2}[(\nabla\theta)^2 + \sin^2\theta(\nabla\varphi)^2] - \mathcal{D}(\theta)\left(\frac{\partial\varphi}{\partial t}\right) - \frac{1}{\alpha}W(\theta)\right\}. \tag{1}$$

Here α is the constant of the nonuniform exchange, c is the spin-wave velocity, M_0 is the saturation magnetization, a is the lattice constant, and $W(\theta)$ is the energy of the relativistic interactions, including anisotropy and Zeeman energy of vector $\mathbf{n}$ in an external field (we consider pure uniaxial magnetic models without anisotropy in the basis plane xy and the field $\mathbf{H}$ is directed along the selected axis).

Let us choose the anisotropy energy in a form $W_a(\theta) = 1/2 \cdot \beta\sin^2\theta$ for the easy-axis case and $W_a(\theta) = 1/2 \cdot \beta\cos^2\theta$ for the easy-plane one, where β is the anisotropy constant.

The form of the function $\mathcal{D}(\theta)$ and Lagrangian parameters are determined by the type of the magnet.

For the analysis of the FM it is necessary to pass the formal limit $c \to \infty$ and put

$$\mathcal{D}(\theta) = -\frac{1-\cos\theta}{\alpha g M_0}, \qquad W(\theta) = W_a(\theta) - \beta h\cos\theta, \qquad h = \frac{H}{H_a},\ H_a \triangleq \beta M_0. \tag{2}$$

When describing an AFM on the basis of equation (1) we suppose $H \ll H_e$, where H_e is so-called exchange field; it is determined as such a field which destroys the AFM ordering. In the scope of the model (1) the magnetization unit vector $\mathbf{m}$ is given by

$$\mathbf{m} = \frac{2}{\delta}\{\mathbf{h} - \mathbf{l}(\mathbf{h}, \mathbf{l})\} + \frac{2}{g\delta M_0}\left[\frac{\partial\mathbf{l}}{\partial t} \times \mathbf{l}\right], \tag{3}$$

where $h \triangleq H/M_0$ is the external magnetic field, see [7, 8]. The functions $\mathcal{D}(\theta)$ and $W(\theta)$ are determined by

$$\mathcal{D}(\theta) = \frac{gH}{c^2}\sin^2\theta, \qquad W(\theta) = W_a(\theta) + \cos^2\theta \cdot H^2/2\beta H_0^2, \tag{4}$$

where $H_0 = c/gl_0$ is a typical field, and $l_0 = \sqrt{(\alpha/\beta)}$ is a typical length coinciding with the width of 1D π-kink. One can note that in the case $H = 0$ the function $\mathcal{D}(\theta) = 0$, so the dynamics of the vector $\mathbf{l}$ has the Lorentz-invariant (LI) properties.

Note that the Lagrangian (1) can be used for the description of a compensated ferrimagnet, i.e. for the magnet with nonequivalent sublattices, but equal magnetizations [26]. For such a magnet $\mathcal{D}(\theta)$-function is given by [27, 28]

$$\mathcal{D}(\theta) = \sin^2\theta\left(\frac{gH}{c^2} + \mathbf{d}\cos\theta\right), \qquad W(\theta) = W_a(\theta) - \beta h\cos\theta. \tag{5}$$

In the dissipationless limit the system has such integrals of motion as the magnetic energy

$$E = \alpha a M_0^2 \int d^2x \left\{ \frac{1}{2c^2}\left[\left(\frac{\partial\theta}{\partial t}\right)^2 + \sin^2\theta\left(\frac{\partial\varphi}{\partial t}\right)^2\right] + \tfrac{1}{2}[(\nabla\theta)^2 + \sin^2\theta(\nabla\varphi)^2] + (1/\alpha)W(\theta)\right\}. \tag{6}$$

and the momentum of magnetization field $\mathbf{P} = \mathbf{P}_{LI} + \mathbf{P}_g$,

$$\mathbf{P}_{LI} = -\frac{\alpha a M_0^2}{c^2}\int d^2x\left[\nabla\theta\frac{\partial\theta}{\partial t} + \nabla\varphi\frac{\partial\varphi}{\partial t}\sin^2\theta\right],$$

$$\mathbf{P}_g = \alpha a M_0^2 \int d^2x \mathcal{D}(\theta)\nabla\varphi, \tag{7}$$

where $\mathbf{P}_{LI}$ is a customary LI term, and $\mathbf{P}_g$ is caused by gyrotropical term with $\mathcal{D}(\theta)$ and is specified due to $\mathbf{P}_g^{(0)}$, which does not contain derivative with respect to time. For the solitonical solution $\theta(\mathbf{r} - \mathbf{v}t)$, $\varphi(\mathbf{r} - \mathbf{v}t)$ it is finite when $\mathbf{v} \to 0$. The value $d\mathbf{P}_g^{(0)}/dt$ determines the gyrotropical force acting on the vortex [29].

4. SOLITONICAL SOLUTIONS

The structure of the soliton is determined by equations for θ and φ following from (1). For the motionless soliton in the pure uniaxial case the solution has the form

$$\varphi = \varphi_0 + \nu\chi, \qquad \theta = \theta^{(0)}(r), \tag{8}$$

where $\varphi_0 = \text{const}$, χ and r are the polar coordinates in the magnet plane xy and $\nu = \pm 1, \pm 2, \ldots$ determines the soliton topological charge (vorticity). The function $\theta^{(0)}(r)$ is a solution of the ordinary differential equation

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\cdot\frac{d\theta}{dr} - \frac{\nu^2}{r^2}\sin\theta\cos\theta + \frac{1}{\alpha}\frac{\partial W}{\partial\theta} = 0. \tag{9}$$

The boundary conditions are determined by the nonsingularity of the solution in the soliton centre ($\theta \to m\pi$ when $r \to 0$, $m = \pm 1, \pm 2, \ldots$) and by the attainment of equilibrium in the distance of the soliton ($\theta = 0$ for the easy-axis case and $\theta = \pi/2$ for the easy-plane one), see [6].

Dynamical solitonical solutions are given by

$$\Box\theta - \sin\theta\cos\theta\left[(\nabla\varphi)^2 - \frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2\right] - \mathcal{F}(\theta)\frac{\partial\varphi}{\partial t} - \frac{1}{\alpha}\frac{\partial W}{\partial\theta} = 0$$

$$\nabla(\sin^2\theta\nabla\varphi) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(\sin^2\theta\frac{\partial\varphi}{\partial t}\right) + \mathcal{F}(\theta)\frac{\partial\theta}{\partial t} = 0, \tag{10}$$

where we used the function $\mathcal{F}(\theta) \triangleq \partial\mathcal{D}/\partial\theta$, $\Box \triangleq \nabla^2 - (1/c^2)\partial^2/\partial t^2$.

Let us take account of dissipation by modifying the Lagrange equations $\delta\mathcal{L}/\delta\theta = 0$, $\delta\mathcal{L}/\delta\varphi = 0$ to $\delta\mathcal{L}/\delta\theta - \delta\mathbf{Q}/\delta\dot{\theta} = 0$, $\delta\mathcal{L}/\delta\varphi - \delta\mathbf{Q}/\delta\dot{\varphi} = 0$, where the dissipation function $\mathbf{Q}$ is

quadratic in time derivative of angular variables θ, φ (and their gradients) with respect to time. The rate dE/dt of the energy dissipation is defined as $2\mathbf{Q}$, where $\mathbf{Q}$ is calculated on the soliton solution. At small soliton velocities $\mathbf{v}$ we can use motionless solutions (8, 9). In the main approach in $\mathbf{v}$ we obtain the expression for the viscous friction force

$$\mathbf{F}_{\text{dis}} = -\eta\mathbf{v}, \qquad \eta = \frac{2\mathbf{Q}}{v^2}, \tag{11}$$

where η is a viscous coefficient, which does not depend on v when $v \to 0$.

Usually the description of magnetic relaxation processes is made on the basis of dissipation function in Gilbert's form with a density $(\partial n/\partial t)^2$. However, in recent years Bar'yakhtar's works [30–33] on the basis of dynamical symmetry of the magnetization field show that the dissipation function of the magnet takes on more complex form and contains some terms of different nature, the exchange term $\mathbf{Q}_e$ and the relativistic one $\mathbf{Q}_r$, $\mathbf{Q} = \mathbf{Q}_e + \mathbf{Q}_r = \alpha a M_0^2 \int d^2x(\mathbf{q}_r + \mathbf{q}_e)$.

The form of exchange dissipation function is to guarantee the dynamical symmetry of the exchange interaction, namely the conservation of the total magnetization of the magnet. In the case of small soliton velocity the value of $\mathbf{q}_e$ is given by

$$\mathbf{q}_e = \frac{\lambda_e}{2\omega_0}[(\nabla\dot{\theta} - \dot{\varphi}\nabla\varphi\sin\theta\cos\theta)^2 + (\nabla\dot{\varphi}\sin\theta + \cos\theta(\dot{\varphi}\nabla\theta + \dot{\theta}\nabla\varphi))^2 + \sin^2\theta(\dot{\varphi}\nabla\theta - \dot{\theta}\nabla\varphi)^2], \tag{12}$$

where λ_e is the exchange relaxation constant. Let us note that sufficiently high symmetry of the exchange interaction causes the presence of gradients of variables θ, φ, $\dot{\theta}$, $\dot{\varphi}$ in the dissipation function, so the dissipation is absent when $\mathbf{m}$ is uniformly rotated [30–32].

The relativistic relaxation processes caused by spin–orbit interaction are accounted for by dissipation function $\mathbf{Q}_r$. The symmetry of these interactions is lower than exchange one; it assumes to represent $\mathbf{Q}_r$ as a quadratic form of $\dot{\theta}$, $\dot{\varphi}$ (without any gradients). For the pure uniaxial magnet, i.e. for the case of absence of anisotropy in the basis plane, useful for vortices, this function must to lead to a conservation of z-component of the magnetization. Because of this terms with $\dot{\varphi}^2$ must be absent, see [31, 33]. This argument was presented earlier in the paper by Halperin and Hoenberg [24] in which, starting from the consideration of symmetry on the basis of the Goldstone theorem and Adler principle, equations of the spin density dynamics of a planar magnet were constructed. Note that the forms of $\mathbf{Q}_e$ and $\mathbf{Q}_r$ are checked with calculations of magnon decay coefficient (the dissipation function in Gilbert's form leads to another dependence of the magnon decay coefficient on its quasi-momentum [30–33]). Relaxation terms of such type were used by Pokrovskiĭ and Khveschenko [14] for the description of the vortex ensemble in the FM.

However, the model of a pure uniaxial magnet ignores, e.g. the anisotropy in a basis plane, always existing in a real magnets, or the magnetoelastic relaxation processes. Besides it is interesting to compare these results with data obtained by using Gilbert term. Therefore we will choose the relativistic function in the form

$$\mathbf{q}_r = \frac{\lambda}{2\omega_0 l_0^2}\dot{\theta}^2 + \frac{\lambda_z}{2\omega_0 l_0^2}\sin^2\theta\dot{\varphi}^2. \tag{13}$$

When $\lambda_z = \lambda$ equation (13) transforms to the standard dissipation function of Landau–Lifshitz or Gilbert type. The case $\lambda_z = 0$ corresponds to the pure uniaxial magnet model, see refs [31, 33]. As we will see below results of our analysis substantially depends on the form of dissipation functions, in particular, the case $\lambda_z = 0$ differs principally from any $\lambda_z \neq 0$.

Taking the dissipation function in this form, the viscous coefficient will be given by

$$\eta = \frac{2\pi\alpha a M_0^2}{\omega_0 l_0^2}(\lambda\gamma_1 + \lambda_z\gamma_2 + \lambda_e\gamma_3). \tag{14}$$

The first two terms are of relativistic nature, they determine η_r; the last term is an exchange viscous coefficient η_e

$$\gamma_1 = \tfrac{1}{2}\int r\,dr\,\theta^{(0)\prime 2}, \qquad \gamma_2 = \frac{\nu^2}{2}\int dr\frac{\sin^2\theta^{(0)}}{r},$$
$$\gamma_3 = \frac{l_0^2}{2}\int r\,dr\left[F_1(r)\theta^{(0)\prime 2} + F_2(r)\frac{\nu^2}{r^2}\sin^2\theta^{(0)}\right], \tag{15}$$

the functions $F_1(r)$ and $F_2(r)$ are introduced in Appendix A.

The dynamical soliton solutions of equation (10) can be analysed analytically only at small velocities $\partial\mathbf{q}(t)/\partial t$ of the soliton centre coordinate, even in the dissipationless limit.

To analyse solutions we will use the solitonical perturbation theory in the way represented by Nikiforov and Sonin [34]. Let us seek solutions linearized over the velocity which appear in the condensate background ($\varphi_c = \mathbf{kr}$, where $\mathbf{k}$ determines the gradient of the phase at infinity and have nonzero value only for MV, see Section 5.2). That is why we use the ansatz

$$\theta = \theta^{(0)}(\xi) + \vartheta(\xi), \qquad \varphi = \varphi^{(0)}(\xi) + \varphi_c(\xi) + \sin^{-1}\theta^{(0)}\mu(\xi), \tag{16}$$

where $\boldsymbol{\xi} = \mathbf{r} - \mathbf{q}(t)$.

In view of (16), equation (10) leads to inhomogeneous system of linear equations with respect to ϑ, μ, $\partial\vartheta/\partial t$, and $\partial\mu/\partial t$.

The criterion of compatibility of such equations, see Appendix A, represents the condition of orthogonality of right-hand sides to solutions of the corresponding homogeneous equations. On the basis of this criterion we will obtain the effective equation of motion of the soliton as a quasiparticle. To write down this equation, we must, however, first of all, analyse motionless solutions $\theta^{(0)}$, $\varphi^{(0)}$, by studying the asymptotic form of functions ϑ, μ, and take explicit form of $\mathcal{F}(\theta)$ and $W(\theta)$ into account. Since such analysis differs substantially for localized topological solitons and magnetic vortices different types of solitons will be taken up separately.

5. DYNAMICS OF SOLITONS: EFFECTIVE EQUATIONS OF MOTION

5.1. *Localized topological solitons*

All two-dimensional localized topological solitons show uniform magnetization distribution in the distance of the soliton kern. These solitons are typical for the easy-axis magnets and for the magnets with finite anisotropy in a basic plane, but LTS can appear in pure easy-plane magnets together with the MV. The main problem in the physics of LTSs is connected with the collapse of such static solitons in the scope of standard phenomenological model (1); it follows from well-known Hobard–Derrick theorem [35, 36], see also [6]. However they can be stabilized allowing for some integrals of motion. In this case we have dynamical solitons; e.g. conservation law of n_z in uniaxial magnet stabilizes precessional soliton [6]. Another way to stabilize LTS, moving with small velocity, is to fall outside the scope of the model (1), accounting, for example, higher derivatives [37, 38], discrete models, or invariants in the energy, which are linear with respect to $\partial n_i/\partial x_j$

[27, 39] (such invariants are present in magnets without inversion centres such as Cu-planes in YBaCuO constructions [40]).

It is difficult to analyse properties of the gas of two-parameter precessional solitons, so we will consider models with statically stabilized 2D solitons.

The motionless LTS is described on the basis of equations (8, 9) with boundary conditions $\theta(0) = \pi \cdot m$, $m = 0, \pm 1, \pm 2, \ldots$ and $\theta(\infty) \to 0$, see [6]. Analytical expression for $\theta^{(0)}(r)$ can be constructed only in two limiting cases.

In the case of large radius soliton ($R \gg l_0$) we can approximate the solution by [6]

$$\cos\theta^{(0)} = \tanh((r - R)/l_0).$$

The topological small-radius soliton ($R \ll l_0$) corresponds to asymptotically exact solution [41]

$$\tan\frac{\theta^{(0)}}{2} = \frac{2}{(\nu - 1)!}\left(\frac{R}{2l_0}\right)^{\nu} K_\nu(r/l_0), \tag{17}$$

where K_ν is the Macdonald function. Following estimations [42–44] we can calculate that the radius R of stable soliton is small in comparison with l_0, $r \leqslant R \ll l_0$. In this actual region equation (17) converts into the well-known Belavin–Polyakov instanton solution [45] $\tan(\theta^{(0)}/2) = (R/r)^{|\nu|}$, see also review [46]. The energy of Belavin–Polyakov LTS $E = |\nu| E_{BP}$, where $E_{BP} = 4\pi\alpha a M_0$ does not depend on R.

Let us proceed to consider LTS dynamical properties. In the limit of small soliton centre velocities let us write down the effective equation of motion. Because of soliton localization, dynamical correlations of ϑ and μ have fast-decreasing asymptotics at infinity. Since surface integrals in the right-hand side of equation (A.2) tend to zero and the criterion of solvability of the system for ϑ and μ leads to such effective equation of motion

$$m^* \frac{\partial^2 \mathbf{q}}{\partial t^2} = G\left[\frac{\partial \mathbf{q}}{\partial t} \times \mathbf{e}_z\right] - \eta \frac{\partial \mathbf{q}}{\partial t}. \tag{18}$$

(see also Appendix B). Here m^* denotes an effective LTS mass; in the dissipationless limit it is given by

$$m^* = \frac{E_0}{c^2} - \pi\alpha a M_0^2 \int r\,\mathrm{d}r \frac{\mathcal{F}(\theta^{(0)})}{\sin\theta^{(0)}}\left[\theta^{(0)\prime}\mu_1 - \frac{\nu \sin\theta^{(0)}}{r}\vartheta_1\right], \tag{19}$$

where $E_0 = \pi\alpha a M_0^2 \int r\,\mathrm{d}r[\theta^{(0)\prime 2} + (\nu^2/r^2)\sin^2\theta^{(0)}]$ is the energy of the motionless soliton. The first term in (19) coincides with ordinary LI one, and it makes the main contribution m^* for the AFM. However, such a term is absent for the case of the FM, and hence the effective mass is determined by the second term, i.e. by μ_1 and ϑ_1, see below.

For the gyrotropical constant we obtain the simple expression

$$G = 2\pi\nu\alpha a M_0^2(\mathcal{Q}(0) - \mathcal{Q}(\infty)). \tag{20}$$

The viscous coefficient is determined by the same formulae (14, 15).

Now we shall consider some specific models of the magnet. The dissipationless LTS's dynamics in the AFM is described by LI equations [43], therefore as one would expect the mass of the soliton calculated using (19) has the LI form, $m^* = E_0/c^2$. In the presence of the magnetic field $\mathbf{H}^2$, directed along the easy axis, the additional term $\Delta m^* \propto H$ appears LTS effective mass.

Unlike the AFM, there is no LI properties in the FM model, so the effective LTS mass is determined only by the second term in equation (19)

$$m^* = \frac{\pi a M_0}{g} \int r\,\mathrm{d}r\left[\theta^{(0)\prime}\mu_1 - \frac{\nu \sin\theta^{(0)}}{r}\vartheta_1\right]. \tag{21}$$

This value coincides with the one obtained earlier by another way [29], and for the small radius soliton it is given by $m^* \approx 3.4E_0/(2\omega_0 l_0)^2$.

The gyrotropical constant for the case of the ferromagnet $G_{FM} = 4\pi vaM_0/g$. This expression is well-known in the magnetic bubble theory [26].

The gyrotropical term for all LTS both in the AFM and in the compensated ferrimagnet goes to zero. Note this situation is realized even for more complex expression $\mathfrak{D}(\theta, \varphi)$, which can appear in different AFM models with Dzialoshinskiĭ–Moria interaction, see refs [47, 48].

Effective equation (18) in the case $\eta = 0$ describes 'free' soliton motion on the analogy of the Larmor precession with the frequency $\omega_G = G/m^*$ and circle radius $R_G = |\partial q/\partial t|/\omega_G$; such effects are well-known in magnetic bubble theory, see [26].

Section 6.1 will show how off-beat precession LTS's motion is exhibited in solitonical response functions.

5.2. *Magnetic vortices*

The structure of magnetic vortex is determined by the same equation (8, 9). The boundary conditions for nonlocalized solutions are $\theta(0) = \pi(1 - p)/2$ and $\theta(\infty) = \theta_0$, where $p = \pm 1$ determines the second topological charge of the vortex (polarization). Properties of motionless vortices are described in detail in [6, 34]. The value θ_0 defines the equilibrium value of $\mathbf{n}$ in the distance of the vortex kern. For the easy-plane AFM and FM with $H = 0$ it is equal to $\pi/2$. But for the FM when $H \neq 0$ there is a cone phase in which $\cos\theta_0 = h$. In this specific case vortices with different polarizations ($p = +1$ and $p = -1$) have nonequal energies. Moreover, a number of interesting effects in MV dynamics appear, but their discussion lies outside the scope of the paper.

Let us note a characteristic property of all magnetic vortices. The energy of any MV diverges as the logarithm of the vortex area S; within logarithmic accuracy one can obtain from (6) that $E_0 = 1/2 \cdot \pi\alpha a v^2 M_0^2 \ln(S/l_0^2)$.

To construct the solution for the moving vortex is a more complex problem. In particular, there is no exact solution describing the moving vortex in the FM even for the small vortex velocity. It is shown only [34] that the motion of vortex in the FM gives rise to nonzero φ gradient, or spin stream following Nikiforov and Sonin [34], in a distance of the vortex, $\nabla\varphi \to \mathbf{k} = \text{const} \cdot \mathbf{v}$ when $|\mathbf{r}| \to \infty$. Given nonuniformity of φ of any nature there is a vortex motion caused by this definitive connection between $\nabla\varphi$ and $\mathbf{v}$. In particular, it determines the behaviour of motion of vortex ensemble, so the total value of $\nabla\varphi$, created by MV's system near some vortex, determines its velocity. Because of that we will consider vortices on the condensate background, i.e. stationary nonuniform state $\theta = \pi/2$, $\varphi_c = \mathbf{kr}$. Let us explain the condensate conception in detail.

It is of interest to note that the presence of the term $\mathbf{P}_g$ leads to the nonzero momentum $\mathbf{P} \equiv \mathbf{P}_g = \mathbf{k} \cdot \alpha a M_0^2 \int d^2x \mathfrak{D}(\theta)$. The structure of $\mathfrak{D}(\theta)$ causes that $\int d^2x \mathfrak{D}(\theta) = \text{const} \cdot S$ for any $\mathfrak{D}(\theta) \not\equiv 0$. Such behaviour is typical for a superfluid liquid, which is described by a complex order parameter $\Psi = |\psi| \exp(i\varphi)$. The momentum density of the superfluid flow is determined by the well-known expression $\mathbf{p} = |\psi|^2 \nabla\varphi = \rho_s \mathbf{v}_s$, where ρ_s is the density of the superfluid component, and $\mathbf{v}_s$ is its velocity. The similarity of these expressions make it possible to talk about a fundamental analogy between superfluid systems and easy-plane magnets (this problem was discussed for the case of ferromagnets, see refs [49, 34]). The momentum density can be naturally juxtaposed with the quantity $\rho_s \mathbf{v}_s$, while the energy density corresponds to the quantity $\rho_s \mathbf{v}_s^2/2$. It follows from the above formulae that quantities $\rho_s^{AFM} = \alpha a M_0^2 (gH/c^2)^2 = 4aH^2/c^2\delta$ and $\rho_s^{FM} = a/\alpha g^2$ can be treated as the analogue of the superfluid density ρ_s for the dynamics of the easy-plane AFM and FM

respectively while $\mathbf{v}_s^{AFM} = (c^2/gH)\cdot\mathbf{k}$ and $\mathbf{v}_s^{FM} = -\alpha g M_0 p\cdot\mathbf{k}$ are corresponding velocities. Analysing linearized eqution of motion for φ it can be easily verified that the local velocity $\mathbf{v}_s$ of 'superfluid' motion of a 'condensate' has the usual form of the type $\mathbf{e}_\chi/r$ near the vortex, while it is equal to the MV's velocity $\partial\mathbf{q}/\partial t$ away from the vortex.

We were able to construct the exact solution only for the vortex in the AFM moving with constant velocity. When $H = 0$ the vortex moving with the velocity $\mathbf{v} = v\cdot\mathbf{e}_x$, can be obtained from the static solution by the Lorentz transform: $x \to x' = (x - vt)\times(1 - v^2/c^2)^{-1/2}$, $y \to y$. The energy and momentum of the vortex when $H = 0$ are determined by the LI formulae: $E_{LI}(v) = E_0(1 - v^2/c^2)^{-1/2}$, $\mathbf{P}_{LI}(v) = (\mathbf{v}/c^2)E_{LI}(v)$, where E_0 is the energy of the motionless soliton. Thus, the vortex effective mass in the case where $H = 0$ is proportional to $\ln S$. When $H \neq 0$ there appears an additional term in the angular variable (phase) φ

$$\varphi = \phi_0 + v\arctan(y/x') + \mathbf{k}\cdot\mathbf{r}, \qquad \mathbf{k} = \mathbf{v}\cdot gH/c^2,$$
$$\theta = \theta(\xi'), \qquad \xi' \equiv r'(l_H^{-2} - k^2)^{1/2}, \qquad r'^2 \equiv x'^2 + y^2. \tag{22}$$

Therefore, in the presence of the magnetic field the energy contains the term $\propto H^2v^2S$: $E(v) = E_{LI}(v) + m^*v^2/2$, where $m^* = \rho_s^{AFM}S$ is the condensate mass. Note this term diverges as the magnet area and not as the logarithm of the area. In substance, it determines the mass of the condensate moving together with a vortex.

Let us proceed to the exploration of nonstationary vortex motion on the basis of general criterion (A.2). Unlike LTS's case, the dynamical correction μ to the solution $\varphi^{(0)}$ has singularities at infinity

$$\mu|_{r\to\infty} \propto r\ln r\cdot\left(\frac{1}{c^2}\frac{\partial^2\mathbf{q}}{\partial t^2} + \frac{\lambda_z}{\omega_0}\frac{\partial\mathbf{q}}{\partial t}\right)\mathbf{e}_\chi. \tag{23}$$

To satisfy the small μ requirement we impose some restrictions on the motion. In the first place, for all dissipation terms the dynamics is possible only with constant velocity. It means that for small finite velocity the value $\partial^2\mathbf{q}/\partial t^2$ is small in comparison with $\partial\mathbf{q}/\partial t$. Secondly, there are more strong limitations for some dissipation terms. For example, the motion is pinned ($\partial\mathbf{q}/\partial t = 0$) for any nonzero relaxation constant λ_z. However we will treat the $\lambda_z = 0$ case only here.

Thus, starting from the criterion (A.2), we will obtain the effective equation of MV motion

$$G\left[\frac{\partial\mathbf{q}_i}{\partial t}\times\mathbf{e}_z\right] + \mathbf{F}_{e,i} - \eta\frac{\partial\mathbf{q}_i}{\partial t} = 0. \tag{24}$$

The form of the gyrotropical term for vortices is given by the same general expression (20), which we found for LTSs. However, because of boundary conditions, we obtain $G_{FM} = 2\pi a M_0 g^{-1}\cdot vp$ and $G_{AFM} = -2\pi v\alpha a M_0^2 gH/c^2 = -16v\alpha M_0 H/gH_e$, where H_e is so-called AFM exchange field. For the compensated ferrimagnet the gyroforce goes to zero when $H = 0$, so the **d**-term makes no contribution to the gyroforce as in the LTS case. Let us point out the essential distinction between G_{AFM} and G_{FM}: (i) in the AFM the gyroconstant is proportional to one of two topological charges, viz. vorticity, unlike the product vp in the FM; (ii) unlike the FM, the gyroforce in the AFM is nonzero only at $H \neq 0$, and for the same values of the parameters M_0 and v, it is less than that in the ferromagnet $G_{AFM} = G_{FM}(4H/H_e) \ll G_{FM}$ (let us remind that AFM ordering is destroyed when $H > H_e$; moreover, the approximation $H \ll H_e$ passes through the paper). The small value of G_{AFM} and its vanishing when $H \to 0$ draws a distinction between dynamics of AFM vortices and FM ones, see below.

The presence of $\mathbf{F}_{e,i} = 2\pi v\alpha a M_0^2 \cdot [\mathbf{k}_i \times \mathbf{e}_z]$ is connected with the constant gradient of the phase, equal to $\mathbf{k}$. Such a term is absent in the case of LTSs. For vortices it can be caused by the spin stream or by external force, such as the vortex interaction force. Setting that $\nabla\varphi$ value at a given vortex is produced by the rest of the vortices with coordinates $\mathbf{q}_i$ one can obtain the expression for $\mathbf{k}$

$$\mathbf{k} = \sum_i \nabla\varphi^{(0)}(\mathbf{r} - \mathbf{q}_i) = \sum_i \nu_i \left[\mathbf{e}_z \times \frac{\mathbf{r} - \mathbf{q}_i}{|\mathbf{r} - \mathbf{q}_i|^2}\right]. \tag{25}$$

Consequently, $\mathbf{F}_{e,i} = -\nabla_i \mathcal{H}_{\text{int}} = e_i \mathbf{E}_i$ describes the interaction between vortices, and the Hamiltonian $\mathcal{H}_{\text{int}} = -2\sum_{i\neq j} e_i e_j \ln|\mathbf{q}_i - \mathbf{q}_j|$ is typical for 2D Coulomb interaction, 'electrical charge' $e_i = \nu_i M_0 \sqrt{(\pi\alpha a)}$, and $\mathbf{E}_i$ means the 'electrical field' acting on given charge e_i (vortex) and caused by other vortices.

Let us note that vortex dynamics in the AFM exhibits salient features in comparison with the FM. Unlike the FM, in which any finite $\mathbf{F}_e$ always causes finite velocity, the vortex in the AFM is proportional to $\mathbf{F}_e/H$ and increases without bounds when $H \to 0$. From the physical point of view this situation is not correct because of finite density of vortices in a topological phase, at $T > T_{\text{KT}}$; such vortices cause nonuniform φ distribution. Therefore, when $H = 0$ or at small fields, AFM vortex ensemble dynamics can be analysed only by using dissipation terms (unlike papers [13–22] in which dissipation processes were ignored in final formulae).

The information about vortex motion effects obtained here will be used in Section 6.2 for the calculation of vortex contribution to the response functions of magnets.

6. SOLITONICAL RESPONSE FUNCTIONS OF 2D MAGNETS

6.1. *Localized topological solitons*

Majority of methods of soliton examination in 1D magnets are connected with analysis of dynamical response functions which are studied by neutron-scattering measurements and besides by decay of electron and nuclear spin waves, see [1, 2]. These methods also were used for the analysis of 2D magnets [3, 4, 15]. For the investigation of 2D magnetic solitons an important role was played by a work by Waldner [50], in which LTS's contribution to the spectrum line width of electron spin waves was isolated. A number of papers study the central peak (CP) of dynamical structure factor (DSF) of easy-plane 2D magnets caused by magnetic vortex gas at $T > T_{\text{KT}}$, see Section 6.2 for more details.

Let us discuss a calculation technique of experimentally observable dynamical characteristics of soliton gas.

It is known that response functions of the magnetic spin system can be determined via Fourier-image of two-spin correlation function

$$S_{ij}(\mathbf{q}, \omega) = \int_{-\infty}^{+\infty} dt \int d^2x e^{i\mathbf{q}\mathbf{r} - i\omega t} \langle S_i(\mathbf{r}, t) S_j(0, 0)\rangle_T, \tag{26}$$

where $\langle \ldots \rangle_T$ denotes thermodynamical averaging, $\mathbf{S}$ is a unit vector along spin direction at $\mathbf{r}$. For the FM S_{ij} is proportional to Fourier-image of $\mathbf{m}$-correlations. For the AFM $\langle S_i S_j\rangle$ can be expressed through correlators of magnetization vector $\mathbf{m}$ and sublattice magnetization vector $\mathbf{l}$. Treating $|\mathbf{m}| \ll |\mathbf{l}| \approx 1$ one can obtain that the correlator $\langle l_i l_j\rangle$ makes the main contribution to $S_{ij}(\mathbf{q}, \omega)$. Long-wave excitations of $\mathbf{l}$-field ($|\nabla\mathbf{l}| \ll r/a$) correspond to quasi-antiphased spin oscillations of neighbour sublattices, hence Fourier components of correlation functions $\langle S_i S_j\rangle$ and $\langle l_i l_j\rangle$ are connected by

$$\langle l_i l_j\rangle_{\mathbf{q}} = \langle S_i S_j\rangle_{\mathbf{q}+\mathbf{K}^0}, \tag{27}$$

where position $\mathbf{K}^0 = (\pi/a, \pi/a)$ determines AFM Bragg peak, see for more details ref. [20]. Correlations of **m**-vector make contribution to response functions. It can be easily expressed via correlations of **l** and $\partial\mathbf{l}/\partial t$ by using equation (3).

Thus, soliton gas contribution to dynamical response functions of 2D magnets is displayed in terms of correlators of dynamical variable $\langle n_i(\mathbf{r}, t)n_j(0, 0)\rangle$. (Note that the dynamical variable represents the unit vector **m** for the FM and **l**-vector for the AFM case).

To calculate such correlations in LTS's case we use, as usual, an approach of dilute soliton gas, which is suitable at low temperatures $T \ll T_c$, T_c is the Curie temperature. Estimating the energy of small radius soliton as $E_{BP} \approx T_c$ (see Section 5.1) one can obtain exponential soliton density $n_s \propto \exp(-E_{BP}/T)$. In our approach of dilute gas N-soliton state can be approximately given by

$$n_{x,y}(\mathbf{r}, t) = \sum_{a=1}^{N} n_{x,y}^{(0)}(\mathbf{r} - \mathbf{q}_a(t)),$$

$$n_z(\mathbf{r}, t) = 1 + \sum_{a=1}^{N} [n_z^{(0)}(\mathbf{r} - \mathbf{q}_a(t)) - 1], \tag{28}$$

where $n_i^{(0)}$, $i = x, y, z$, determines isolated soliton centred about $\mathbf{q}_a(t)$. Note that such expressions can be used for locally sensitive out-of-plane **n**-correlations in easy-plane case, but globally sensitive in-plane correlations show more complete situation (see the next section).

In our approach of dilute gas the main contribution to correlation function with respect to n_s is produced by correlations between a given soliton (or a vortex) in different instants of time, so solitonical contribution to $S_{ij}(\mathbf{q}, \omega)$ can be written down as $n_s|f_{ij}(\mathbf{q})|^2 F(\mathbf{q}, \omega)$, where the form factor $f_{if}(\mathbf{q})$ is approximated only by statical structure. For example, $f_{zz}(\mathbf{q}) = \int d^2x(1 - n_z^{(0)}(\mathbf{r}))\exp(i\mathbf{qr})$. Properties of dynamical response functions are determined by

$$F(\mathbf{q}, \omega) = \int_{-\infty}^{+\infty} dt \exp(-i\omega t)\langle \exp(i\mathbf{q}[\mathbf{r}(t) - \mathbf{r}(0)])\rangle_T, \tag{29}$$

which depends on the type of soliton motion, i.e. on the form of $\mathbf{r}(t)$. Calculation of $F(\mathbf{q}, \omega)$ often uses free soliton approximation, neglecting soliton collision and thermostat effects such as magnon and phonon action, in particular, viscosity. Unless gyroeffects are also present (constant $G = 0$ in equation (18) for the soliton centre coordinate), $\mathbf{r}(t) = \mathbf{r}(0) + \mathbf{v}t$, $\mathbf{v} = \text{const.}$ as in the case of 1D solitons. Therefore thermodynamical averaging in (26) means averaging with respect to soliton velocity distribution, $\langle(\ldots)\rangle_T = \int d\mathbf{v} P(v)$, where $P(v)$ is a velocity distribution function. In this simple case it is easy to show that

$$F(\mathbf{q}, \omega) = \pi P(v)|_{v=\omega/q}. \tag{30}$$

In particular, processing from Maxwellian velocity distribution, we obtain a Gaussian CP, centred about $\omega = 0$

$$F_G(\mathbf{q}, \omega) = \left(\frac{2\pi m^*}{Tq^2}\right)^{1/2} \exp\left(-\frac{m^*\omega^2}{2Tq^2}\right), \tag{31}$$

where CP-width $\Gamma_\omega = (2Tq^2/m^*)^{1/2}$ is determined by thermal fluctuations of soliton velocity. It should be noted that solitonical contributions to the DSF CP of such type is well-known both for 1D kinks and for 2D vortices. Gaussian shape is typical for all locally sensitive components of response functions of 1D magnets and 2D easy-plane ones.

In reality it is necessary to account for gyroforce and soliton viscous retardation effects. Given gyroforce 'free' soliton motion in analogy with Larmor precession with the frequency $\omega_G = G/m^*$; such effects are well-known in magnetic bubble theory, see [26]. When $G = 0$, but $\eta \neq 0$ equation (18) describes the soliton retardation as $\mathbf{v}(t) = \mathbf{v}(0)\exp(-\omega_r t)$, where $\omega_r = m^*/\eta$ is the relaxation frequency. It is understood that real r.m.s. soliton velocity does not depend on time, but has a constant value $\bar{u} = v_T = \sqrt{(2T/m^*)}$, where T is the temperature. Thermal soliton motion can be accounted by additional random force $\boldsymbol{\zeta}(t)$ such that $\langle \boldsymbol{\zeta}(t) \rangle = 0$, but $\langle \zeta_i(t)\zeta_j(0) \rangle_T = 2T\eta\delta_{ij}\delta(t)$, $i, j = x, y$. Now our equation of soliton motion has the form of Fokker–Planck equation for the particle moving in x, y-plane, or for the charge in the magnetic field when $G \neq 0$. In the scope of this approach, which is widely used for 1D solitons, see review articles [1, 2, 51] we can calculate $F(\mathbf{q}, \omega)$ and spin correlators for any relation between these parameters, such as viscosity η, gyroforce G, effective mass m^*, and temperature T. Note that the problem of average spin gas velocity is more complex for the case of nonlocalized solitons, i.e. magnetic vortices. We will show below that average MV's velocity is caused not by thermal fluctuations, but by vortex–vortex interaction due to effects of vortex 'freezing into condensate', see Section 6.2.

Response functions for LTSs in 2D magnet were calculated by Ivanov and Kolezhuk [52]. Following this article, let us present the results of response function calculation for any soliton motion. It has been shown in Section 5.1 that the gyroconstant $G = 0$ for the localized soliton in the AFM and in compensated ferrimagnet (note that $G \neq 0$ for vortices in easy-plane AFM). In our case of arbitrary motion CP shape is determined by the parameter ql_r, where $l_r = v_T/\omega_r = \sqrt{(Tm^*)}/\eta$ is the mean free path. Under condition $ql_r \gg 1$, we obtain the Gaussian CP (31) typical for a free soliton motion, but when $ql_r \ll 1$ it should be realized a Lorentzian peak

$$F_{\mathrm{L}}(\mathbf{q}, \omega) = \frac{2Dq^2}{\omega^2 + (Dq^2)^2}. \tag{32}$$

Here $D = T/\eta$ is the soliton gas diffusion constant. For the FM the value $\omega_G = G/m^*$ is about the frequency of linear magnons for small quasi-momenta, so gyroeffects can be substantial. The analysis shows that the structure of $f(\mathbf{q}, \omega)$ depends on ql_r and $\omega_G\tau_r$, $\tau_r \triangleq 1/\omega_r$. When $\omega_G\tau_r \ll 1$ Larmor circles do not form and gyroeffects have no display. Therefore we have a solitary CP of the Lorentzian shape (32) when $ql_r \ll 1$ or of Gaussian shape (31) when $ql_r \gg 1$.

The gyrotropical motion, $\omega_G\tau_r \gg 1$, shows more interesting properties. In this case there are a number of additional peaks at $\omega = n\omega_G$ for any nonzero $\mathbf{q}$. DSF $F(\mathbf{q}, \omega)$ can be written as an unwieldy expression [52], so we will present its asymptotics only.

There is a single peak with $n = 0$ (31) when $q \ll q_G$. Intensities of additional peaks increase when q increases and for $q_G \ll q \ll q_G(\omega_G\tau_r)^{1/2}$ the DSF $F(\mathbf{q}, \omega)$ contains a number of Lorentzian peaks

$$F(\mathbf{q}, \omega) = \sum_{n=-\infty}^{\infty} \exp(-\Gamma_\omega\tau_r) F_n(\Gamma_\omega\tau_r) \frac{2\Gamma_\omega}{\Gamma_\omega^2 + (\omega - n\omega_G)^2}, \tag{33}$$

with widths $\Gamma_\omega = Dq^2/(\omega_G\tau_r)^2$ and intensities which depend on n weakly. Such patterns correspond to 'free' rotation soliton motion due to gyroforce when $\tau_r \to \infty$. Because $\Gamma_\omega \ll \omega_G$ all peaks (33) are very narrow.

When q increases peak widths increase too because of viscosity and when $\Gamma > \omega_G$, i.e. $q > q_G(\omega_G\tau_r)^{1/2}$ we have a general Gaussian peak (31).

Estimations of q_r, τ_r, q_G, and ω_G by [52] show that all these types of motion can be

realized in 2D magnets. Unfortunately, such experiments have not been carried out to our knowledge.

6.2. *Magnetic vortices*

Thermodynamics of magnetic vortices have been explored by a number of theoretical and experimental papers [3, 4, 15–22]. The variety of nonlinear properties of magnets connected with 2D solitons show substantial difference between LTSs and MVs. Some differences in dynamical properties have been demonstrated in Section 5; they are connected with 'condensate' effects for vortices and with stationary-stable MV's dynamics, i.e. with the absence of inertial term in the effective equations of motion (24).

The last section mentioned the unusual character of vortex contribution to the magnet thermodynamics too. These distinctions are caused by the nature of vortices, namely by their nonlocalizability, and what in-plane DSF components are globally sensitive to the presence of vortices. Unlike LTSs such solitons undergo strong influence due to other vortices through 2D Coloumb interaction. Thermal fluctuations of such interaction give rise to nonzero vortex gas average velocity (Let us note that ordinary expression for LTS's r.m.s. velocity $\bar{u} = \sqrt{(2T/m^*)}$ and standard Maxwellian distribution are unsuitable for the vortex case due to the absence of the inertial term in equations of motion).

Before the calculation of vortex contribution to response functions we want to attract attention to the problem of vortex existence from the thermodynamical point of view. The point is that the vortex energy diverges with the logarithm of area and vortex appearance advantage is not trivial. The presence of vortices is connected with a Kosterlitz–Thouless (KT) type of topological phase transition. Berezinskiĭ [53], and Kosterlitz and Thouless [12] show that in easy-plane magnets below critical temperature T_{KT}, in the so-called Berezhinskiĭ phase, there are vortex–antivortex binding pairs, i.e. localized solitons. Such pairs begin to unbind above T_{KT}, in the topological phase. Following [12] let us estimate the value of T_{KT}. Free energy variation caused by the creation of one vortex is given by $\Delta F = E_0 - T\mathcal{S}$, where $E_0 = 1/2 \cdot \pi\alpha a v^2 M_0^2 \ln(S/l_0^2)$ is the vortex energy, $\mathcal{S} = \ln S$ is the entropy of indeterminacy of vortex position. Hence, vortex can appear above the critical temperature $T_{KT} \approx 1/2 \cdot \pi\alpha a v^2 M_0^2$ due to $(T_{KT} - T)\ln S < 0$. More accurate calculations [12] lead to $T_{KT} \approx 0.5\pi^2 \alpha a v^2 M_0^2$.

Thus, there is a dilute vortex gas in the topological phase. The presence of MVs displays spin correlations in the finite distance $\langle n(\mathbf{r})n(0)\rangle \propto \exp(-r/\xi)$, where the correlation length is

$$\xi(T) = \xi_0 \exp(b\tau^{-1/2}), \qquad \tau \triangleq (T - T_{KT})/T_{KT}, \tag{34}$$

ξ_0 and b are constants [12, 54, 50].

Vortex gas density n_v tends to zero when $T \to T_{KT}$ and increases when T rises, but its calculation *ab origin* stands in need of the account of vortex gas interaction. For the moment such analysis has not been carried out to our knowledge. To estimate n_v a simple relation $n_v \approx (2\xi)^{-2}$ is used, see [54, 55].

Let us proceed to the calculation of vortex contribution to the DSF. As in the LTS's case we will approximate N-vortex solution

$$\varphi(\mathbf{r}, t) = \sum_{a=1}^{N} \varphi^{(0)}(\mathbf{r} - \mathbf{q}_a(t)),$$

$$n_z(\mathbf{r}, t) = \sum_{a=1}^{N} n_z^{(0)}(\mathbf{r} - \mathbf{q}_a(t)). \tag{35}$$

For the dissipationless model of the easy-plane FM calculations were made by Mertens *et al.* [15]. The main relationships are as follows. Out-of-plane DSF components are locally sensitive to the presence of vortices due to the localization of $n_z^{(0)}$ near vortex kern. There is a Gaussian CP, which intensity increases $\propto n_v$ and depend on the form factor (as one can see such behaviour was demonstrated in Section 6.1 for localized solitons). Unlike out-of-plane correlations all in-plane DSF components, are globally sensitive to the presence of vortices, which act to break long-range ordering in the magnet's plane. The central peak of Lorentzian shape does not depend on vortex form factor, i.e. **n**-distribution near vortex centre. The intensity of such a CP decreases when vortex density diminishes, because **n**-distribution in the magnets becomes uniform on the average.

The calculation on the basis of N-vortex solution (35) leads to the expression like (30) for the LTSs, but with velocity distribution of another nature. It was mentioned above that $P(v)$ in the MV case is caused by strong Coulomb interaction. Following Huber [13], let us introduce the self-consistent effective 'electric field' **E**, describing interaction with other vortices. There is a formal similarity between the equation of vortex motion and the equation of motion of guiding centre in a 2D plasma, which lies in a perpendicular magnetic field [13]. It allows to estimate the value of $\langle E^2 \rangle$ by results obtained by Taylor and McNamara [56] $\langle E^2 \rangle = n_v \pi e^2 \ln \Lambda$, where n_v is the equilibrium vortex density, and Λ determines the proximity to the thermal equilibrium. For the cases of a random and a thermal distributions Λ is expressed by $\Lambda_R = S/\pi a^2$ and by $\Lambda_T = 4\pi^2 T_{KT}/n_v e^2 a^2$ respectively.

Using this $\langle E^2 \rangle$ and equation (24) we obtain the value of the r.m.s. vortex velocity

$$\bar{u} = \frac{e\langle E^2 \rangle^{1/2}}{(G^2 + \eta^2)^{1/2}}. \tag{36}$$

For the FM it gives a simple expression

$$\frac{\bar{u}_{FM}}{\bar{u}_H} = (1 + (H_r^{FM}/H_a)^2)^{-1/2}, \tag{37}$$

where

$$\bar{u}_H = \frac{gM_0}{2}(\alpha\beta)^{1/2}(\pi n_v l_0^2 \cdot \ln \Lambda)^{1/2} \tag{38}$$

is a typical r.m.s. velocity in the ferromagnet, obtained by Huber [13] for the Λ_T case, $H_r^{FM} = \eta(gM_0\sqrt{(\beta)}/2\pi a)$ is a typical field, which determines the influence of viscosity. Estimating η as in ref. [14] we obtain that $H_r^{FM}/H_a \ll 1$, and hence the Huber expression (38) is a good approximation for the FM case.

Unlike the FM, we have an opposite situation for the AFM connected with the vanishing of G as $H \to 0$. Using concrete expressions for G_{AFM} and η we obtain

$$\bar{u}_{AFM} = \frac{c}{2}\left(\frac{\pi n_v l_0^2 H_0^2}{H^2 + H_r^2} \ln \Lambda\right)^{1/2}, \tag{39}$$

where for convenience we use the typical fields H_0 (see above) and $H_r \triangleq H_r^{AFM} = \eta(g\delta/8\pi a)$, H_r^{AFM} is proportional to the relaxation coefficient. Using for η the above mentioned formulae (15) and estimations by ref. [47], we obtain $H_r^{AFM} \approx 0.05 H_0$ when $T \approx T_{KT}$. Thus, the value $\bar{u}_{AFM}/\bar{u}_H \approx H_e/4H$ is seen to be inversely proportional to H, but for $H \ll H_r$, $\bar{u}_{AFM}/\bar{u}_H$ does not depend on H. Rather, $\bar{u}_{AFM}/\bar{u}_H \approx H_e/4H_r^{AFM}$. Therefore the r.m.s. velocity in the AFM is greater than in the ferromagnet. These results agree in kind with data simulated by Völkel *et al.* [21]. It would be interesting to check the dependence of $\bar{u}_{AFM} \propto 1/H$, but numerical simulations for $H \neq 0$ have not been carried

out, as far as we know. In the case where $H < H_r^{AFM}$ the value $\bar{u}$ does not depend on H, but it is inversely proportional to relaxation constant η. Since the relaxation constant λ and, consequently, the viscous coefficient increase when T increases, $\eta \propto T^n$, $n = 2$ for the FM [14] and $n = 3$ for the AFM [57], this formula describes a decrease of the value of $\bar{u}$ with increasing T stronger than in the FM. Note this results agree with computer simulation in the vicinity of T_{KT} when T increases stronger than for the data [21, 22].

Now we proceed to the concrete DSF calculation. As in the case of localized solitons we have a Gaussian CP for out-of-plane correlations. However, unlike Maxwellian $P(v)$ distribution with $\bar{u} = \sqrt{(2T/m^*)}$, we have a Gaussian vortex velocity distribution with $\bar{u}$ like (36)

$$F_G(\mathbf{q}, \omega) = \frac{1}{2\pi^{3/2} q\bar{u}} \exp[-(\omega/q\bar{u})^2], \tag{40}$$

Thus, out-of-plane correlations of the FM are given by

$$S_{zz}^{FM}(\mathbf{q}, \omega) = n_v |f^{FM}(q)|^2 F_G(\mathbf{q}, \omega) \tag{41}$$

where $f^{FM}(q) = \int d^2x \cos^2 \theta^{(0)} \exp(i\mathbf{qr})$ is a vortex form factor.

Now we will consider AFM out-of-plane correlations. It was mentioned in the last section that in the AFM case there are two different contributions of terms **m** and **l**. As a result one can obtain

$$S_{zz}^{AFM}(\mathbf{q}, \omega) = |f_1^{AFM}(\mathbf{q})|^2 F_G(\mathbf{K}^0 - \mathbf{q}, \omega) + \frac{4h^2}{\delta^2} |f_2^{AFM}(\mathbf{q})|^2 F_G(\mathbf{q}, \omega), \tag{42}$$

where $f_k^{AFM}(q) = \int d^2 \cos^k \theta^{(0)} \exp(i\mathbf{qr})$ are two different vortex form factors which determine distributions of **l** and **m** respectively. Let us note that the intensity of the third term $\propto H^2$. The analysis of this dependence can be a good test for comparison of experimental and theoretical data.

The problem of in-plain correlations is more complex. It was analysed by Mertens *et al.* [15] by using the analogy with the problem of π-kink motion via the FM [58]. After tedious calculations we will obtain that $S_{xx}^{FM}(\mathbf{q}, \omega) = F_L(\mathbf{q}, \omega)$, see ref. [15], and $S_{xx}^{AFM}(\mathbf{q}, \omega) = F_L(\mathbf{K}^0 - \mathbf{q}, \omega)$, where

$$F_L(\mathbf{q}, \omega) = \frac{\kappa^3}{8\pi n_v (\omega^2 + \kappa^2[1 + (q/2n_v^{1/2})^2])^2} \tag{43}$$

determines peak of (squared) Lorentzian shape, $\kappa = \bar{u}(\pi n_v)^{1/2}$.

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APPENDIX A

Linearized equations of motion of 2D solitons

In view of the criterion (16) the system (10) transforms to the form

$$(\hat{L}_1 + F_1)\vartheta - \hat{L}_2\mu - \hat{R}_1\frac{\partial\vartheta}{\partial t} - \hat{R}_2\frac{\partial\mu}{\partial t} = f_1,$$

$$(\hat{L}_1 + F_2 - F_3)\mu + \hat{L}_2\vartheta - \hat{R}_3\frac{\partial\mu}{\partial t} + \hat{R}_2\frac{\partial\vartheta}{\partial t} = f_2, \tag{A.1}$$

where we use following symbols

$$\hat{L}_1 = \nabla^2 - \nu^2/r^2, \qquad \hat{L}_2 = 2\cos\theta^{(0)}\nabla\varphi^{(0)}\nabla,$$

$$F_1 = \frac{2\nu^2}{r^2}\sin^2\theta^{(0)} - \frac{1}{\alpha}\frac{\partial^2 W}{\partial\theta^{(0)2}}, \qquad \hat{R}_1 = \frac{\lambda}{\omega_0 l_0^2} + \frac{\lambda_e}{\omega_0}\hat{L}_1,$$

$$F_2 = 2\theta^{(0)\prime 2} - \cot\theta^{(0)}\frac{1}{\alpha}\frac{\partial W}{\partial\theta^{(0)}}, \qquad \hat{R}_2 = \frac{\lambda_e}{\omega_0}\hat{L}_2 + \frac{\mathcal{F}(\theta^{(0)})}{\sin\theta^{(0)}},$$

$$F_3 = \theta^{(0)\prime 2} - \frac{\nu^2}{r^2}\sin^2\theta^{(0)}, \qquad \hat{R}_3 = \frac{\lambda_e}{\omega_0}\hat{L}_1 + \frac{\lambda_z}{\omega_0 l_0^2} + \frac{\lambda_e}{\omega_0}F_3$$

$$f_1 = -\frac{1}{c^2}\frac{\partial^2\mathbf{q}}{\partial t^2}\nabla\theta^{(0)} + \sin 2\theta^{(0)}\left(\mathbf{k} - \frac{\mathcal{F}(\theta^{(0)})}{\sin 2\theta^{(0)}}\frac{\partial\mathbf{q}}{\partial t}\right)\nabla\varphi^{(0)}$$
$$- \frac{\nabla\theta^{(0)}}{\omega_0}\frac{\partial\mathbf{q}}{\partial t}\left(\frac{\lambda}{l_0^2} + \lambda_e F_1\right),$$

$$f_2 = -\frac{\sin\theta^{(0)}}{c^2}\frac{\partial^2\mathbf{q}}{\partial t^2}\nabla\varphi^{(0)} - \cos\theta^{(0)}\left(\mathbf{k} - \frac{\mathcal{F}(\theta^{(0)})}{\sin 2\theta^{(0)}}\frac{\partial\mathbf{q}}{\partial t}\right)\nabla\theta^{(0)}$$
$$- \frac{\sin\theta^{(0)}\nabla\varphi^{(0)}}{\omega_0}\frac{\partial\mathbf{q}}{\partial t}\left(\frac{\lambda_z}{l_0^2} + \lambda_e F_2\right).$$

The criterion of solvability of the system (A.1) can be written in the form

$$\langle f_1\vartheta_0 + f_2\mu_0\rangle + \left\{(\vartheta_0\hat{R}_1 - \mu_0\hat{R}_2)\frac{\partial\vartheta}{\partial t} + (\vartheta_0\hat{R}_2 + \mu_0\hat{R}_3)\frac{\partial\mu}{\partial t}\right\}$$
$$= \langle\nabla\cdot(\vartheta_0\nabla\vartheta - \vartheta\nabla\vartheta_0 + \mu_0\nabla\mu - \mu\nabla\mu_0 + 2\cos\theta^{(0)}\nabla\varphi^{(0)}[\vartheta\mu_0 - \vartheta_0\mu])\rangle \tag{A.2}$$

Here $\vartheta_0 = (\mathbf{d}, \nabla\theta^{(0)})$ and $\mu_0 = \sin\theta^{(0)}(\mathbf{d}, \nabla\varphi^{(0)})$ denote solutions of the corresponding homogeneous equations, see [34], and $\mathbf{d}$ is an arbitrary translation vector.

APPENDIX B

Effective mass of localized topological soliton

Since the velocity and the acceleration of soliton centre are connected by equation (A.1), we will find the solution of (A.1) in the form

$$\vartheta = \vartheta_1 \frac{\partial \mathbf{q}}{\partial t} \mathbf{e}_\chi + \vartheta_2 \frac{\partial \mathbf{q}}{\partial t} \mathbf{e}_r, \qquad \mu = \mu_1 \frac{\partial \mathbf{q}}{\partial t} \mathbf{e}_r + \mu_2 \frac{\partial \mathbf{q}}{\partial t} \mathbf{e}_\chi. \tag{B.1}$$

A straightforward calculation leads to

$$m^*_{\nu\lambda} \frac{\partial^2 \mathbf{q}_\lambda}{\partial t^2} = G \varepsilon_{\nu\lambda} \frac{\partial \mathbf{q}_\lambda}{\partial t} - \eta \frac{\partial \mathbf{q}_\nu}{\partial t}. \tag{B.2}$$

Here $m^*_{\nu\lambda} = \begin{pmatrix} m^* & \mu^* \\ -\mu^* & m^* \end{pmatrix}$ is a tensor of effective mass of LTS. Its diagonal term has a form

$$m^* = \pi \alpha a M_0^2 \int r \, dr \left\{ \frac{1}{c^2} \left[\theta^{(0)\prime 2} + \frac{\nu^2}{r^2} \sin^2 \theta^{(0)} \right] - \frac{\mathcal{F}(\theta^{(0)})}{\sin \theta^{(0)}} \left[\theta^{(0)\prime} \mu_1 - \frac{\nu}{r} \sin \theta^{(0)} \vartheta_1 \right] \right.$$
$$\left. - \theta^{(0)\prime} \hat{R}_1 \vartheta_2 - \frac{\nu}{r} \sin \theta^{(0)} \hat{R}_3 \mu_2 + \frac{2\nu \cos \theta^{(0)}}{r^2} \frac{\lambda_e}{\omega_0} \left[\frac{\nu}{r} \sin \theta^{(0)} \vartheta_2 + \theta^{(0)\prime} \mu_2 \right] \right\}, \tag{B.3}$$

and nondiagonal ones

$$\mu^* = \pi \alpha a M_0^2 \int r \, dr \left\{ \theta^{(0)\prime} \hat{R}_1 \vartheta_1 - \frac{\nu}{r} \sin \theta^{(0)} \hat{R}_3 \mu_1 + \frac{\mathcal{F}(\theta^{(0)})}{\sin \theta^{(0)}} \left[\theta^{(0)\prime} \mu_2 + \frac{\nu}{r} \sin \theta^{(0)} \vartheta_2 \right] \right.$$
$$\left. - \frac{2\nu \cos \theta^{(0)}}{r^2} \frac{\lambda_e}{\omega_0} \left[\frac{\nu}{r} \sin \theta^{(0)} \vartheta_1 - \theta^{(0)\prime} \mu_1 \right] \right\} \tag{B.4}$$

In the case of small dissipation straightforward calculations finally lead to (18). In the dissipationless limit $\mu^* \equiv 0$, but for m^* one can obtain the expression (19).