INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 4259–4267 PII: S0305-4470(03)59037-0

Deflection of a vortex pair by an interface in easy-plane ferromagnets

J-G Caputo[1,4], J P Zagorodny[2], Yu Gaididei[3] and F G Mertens[2]

[1] Laboratoire de Mathématiques, INSA de Rouen, BP 8, 76131 Mont-Saint-Aignan Cedex, France
[2] Physics Institute, University of Bayreuth, Bayreuth, Germany
[3] Bogoliubov Institute of Theoretical Physics, Academy of Sciences of Ukraine, Kiev, Ukraine
[4] Laboratoire de Physique théorique et modelisation, Université de Cergy-Pontoise and CNRS 5 Mail Gay Lussac, 95031 Cergy-Pontoise Cedex, France

E-mail: caputo@insa-rouen.fr, Juan.Zagorodny@uni-bayreuth.de, yugaid@bitp.kiev.ua and franz.mertens@uni-bayreuth.de

Received 31 January 2003
Published 3 April 2003
Online at stacks.iop.org/JPhysA/36/4259

Abstract

We study the motion of a vortex–antivortex pair in easy-plane ferromagnets crossing an interface between two media with different anisotropy. A simple description based on the Thiele approach is obtained. The collective variables are the vortex centres and core radii, the latter are assumed to be slaved to the former. For a normal crossing of the interface by the vortex pair, a simple estimate of the ratio of the separation distances is obtained from energy conservation. This prediction is validated by direct numerical simulations of the Landau–Lifshitz equations for the anisotropic Heisenberg model, on a spin lattice divided into two regions which have different anisotropies.

PACS numbers: 75.10.Hk, 75.30.Gw, 75.40.Mg

1. Introduction

Vortices, which can be described, e.g. as phase singularities of a complex field, occur in many areas of physics. The most familiar ones occur in hydrodynamics where they have been studied for many years [1, 2]. A more recent example concerns optical vortices [3] which can be described using a Ginzburg–Landau type amplitude equation. The motion of electrons in a superconductor is governed by a similar model [4] and can also display such structures. Finally, 2D magnetic systems described by a Heisenberg spin Hamiltonian are among the first in condensed matter physics where vortices have been studied systematically [5].

The main ingredient to obtain a vortex solution is a field equation that reduces in some limit to a Laplacian yielding the energy density $(\nabla\phi)^2$ where ϕ is the phase. A vortex is then very close to an elementary charge in electrostatics. In 2D the Green's function is $\phi_G = \log(r)$

which gives rise to divergences for both $r \to 0$ and $r \to \infty$. In reality the singularity at $r = 0$ is 'regularized' by an additional short-range field to form a finite vortex core, an example is the density variation in a superfluid model [4]. The logarithmic divergence for large r, however, remains so that a single vortex has infinite energy in an infinite system and therefore cannot move. If, however, it is associated with an antivortex (to form the equivalent of a dipole) then the energy of the pair is finite and motion occurs. Because of its topology such a vortex pair can be naturally formed in a vacuum, in contrast to a single vortex. A good illustration is the vortex ring—the 3D equivalent of a vortex pair—created by an object falling in a liquid [7].

A natural question to ask is what happens to such a vortex pair when it encounters inhomogeneities of the material. Here we consider the case of an interface between two media of different properties. We will illustrate this in the case of a two-dimensional easy-plane ferromagnet, described by the Heisenberg exchange Hamiltonian with anisotropic coupling of the vertical z-components m of the spins. The orientation of the spins can be parametrized by the azimuthal phase ϕ and by m, where m is the field 'regularizing' the vortex core. The radius of the core depends on the anisotropy of the material. This is the property that we will change from one side of the interface to the other.

A simplified model of the motion of a vortex pair in this system is obtained assuming a travelling frozen field where the time dependence enters only through the vortex centres $R_i(t), i = 1, 2$ [6]. In this context, this collective coordinate approach is called the Thiele approximation [8, 9]. Here we extend it to the case of an inhomogeneous material by introducing as an additional variable the core size κ_i of each vortex. To simplify the description we assume this variable to be slaved to the position R_i.

To our knowledge this is a first attempt to describe the motion of a magnetic vortex–antivortex pair across an interface between two materials. To keep things simple we present here only the case of normal incidence of the vortex pair upon the interface. After introducing the general Hamiltonian for the inhomogeneous Heisenberg system, we present the collective coordinate approach in section 2. In section 3 we describe the solution in the case of a vortex pair impinging normally on an interface and compare it to that obtained by direct simulation of the spin equations in section 4. We conclude in section 5.

2. Hamiltonian and equations of motion

We consider the classical two-dimensional Heisenberg easy-plane ferromagnet with spatially inhomogeneous uniaxial anisotropy. On a square lattice the Hamiltonian of the system has the form

$$\mathcal{H} = -\frac{1}{2} J \sum_{(\vec{n},\vec{a})} (\vec{\mathcal{S}}_{\vec{n}} \cdot \vec{\mathcal{S}}_{\vec{n}+\vec{a}}) + \frac{1}{2} \sum_{\vec{n}} \mathcal{K}_{\vec{n}} \left(\mathcal{S}^z_{\vec{n}}\right)^2 \tag{1}$$

where $\vec{\mathcal{S}}_{\vec{n}} \equiv \left(\mathcal{S}^x_{\vec{n}}, \mathcal{S}^y_{\vec{n}}, \mathcal{S}^z_{\vec{n}}\right)$ is the classical spin variable on the site $\vec{n}$, the exchange integral, J, and the anisotropy constant, $\mathcal{K}_{\vec{n}}$, are positive. In the first term $\vec{a}$ is the lattice vector so that the summation runs over nearest-neighbour pairs. The spin motion is governed by the Landau–Lifshitz equations, which in normalized form can be written as

$$\frac{\mathrm{d}\vec{\mathcal{S}}_{\vec{n}}}{\mathrm{d}t} = -\vec{\mathcal{S}}_{\vec{n}} \times \frac{\partial \mathcal{H}}{\partial \vec{\mathcal{S}}_{\vec{n}}} = \vec{\mathcal{S}}_{\vec{n}} \times \left\{ J \sum_{\vec{a}} \vec{\mathcal{S}}_{\vec{n}+\vec{a}} - \mathcal{K}_{\vec{n}} \mathcal{S}^z_{\vec{n}} \hat{z} \right\}. \tag{2}$$

These equations conserve the length of the spins $|\vec{\mathcal{S}}_{\vec{n}}| \equiv \mathcal{S}$, which has units of action for the classical and quantum models.

We shall consider the case of weak anisotropy $\mathcal{K}_{\vec{n}} \ll J$ when the characteristic size of excitations $\sqrt{J/\mathcal{K}}$ is larger than the lattice constant a (= 1). In this case we can use the continuum approximation of the Hamiltonian (1)

$$H = \frac{1}{2}\int \mathrm{d}\vec{r}\{(\partial_x \vec{S})^2 + (\partial_y \vec{S})^2 + K(\vec{r})(S^z)^2\} \tag{3}$$

where $H = \frac{\mathcal{H}-H_0}{J\mathcal{S}^2}$, H_0 is a constant, the spin length has been rescaled so that $\vec{S} = \vec{\mathcal{S}}/\mathcal{S}$, and $K \equiv \frac{\mathcal{K}}{Ja^2} \ll 1$. In this continuum approximation the Landau–Lifshitz equation becomes

$$\frac{1}{J\mathcal{S}}\frac{\mathrm{d}\vec{S}}{\mathrm{d}t} = -\vec{S} \times \frac{\delta H}{\delta \vec{S}} = \vec{S} \times \{\Delta\vec{S} - K(\vec{r})S_z\hat{z}\}. \tag{4}$$

The denominator $J\mathcal{S}$, with units of inverse time, can be absorbed into a dimensionless time variable. Again equation (4) preserves the length of the spins so it is convenient to parametrize the vector $\vec{S}(\vec{r}, t)$ with two variables, the in-plane phase $\phi(\vec{r}, t)$ and the on-site magnetization $m(\vec{r}, t) \equiv S^z(\vec{r}, t)$. The magnetization $m(\vec{r}, t)$ and the phase $\phi(\vec{r}, t)$ satisfy the Hamiltonian equations

$$\dot{\phi} = \frac{\delta H}{\delta m} \qquad \dot{m} = -\frac{\delta H}{\delta \phi} \tag{5}$$

so that they are canonically conjugate. In terms of these variables the Hamiltonian (3) is

$$H = \frac{1}{2}\int \mathrm{d}\vec{r}\left\{\frac{(\nabla m)^2}{1-m^2} + (1-m^2)(\nabla\phi)^2 + K(\vec{r})m^2\right\}. \tag{6}$$

Before describing the collective variable approach for a vortex pair we recall briefly that the structure of a single static vortex in the homogeneous case is the pair of functions $(m, \phi) = (f(r\sqrt{K}), q\Phi(\vec{r}))$ that satisfy equations (5). A vortex has topological charge $q = 1$ while $q = -1$ for an antivortex. Away from the vortex centre (assumed for the moment at $\vec{r} = \vec{R} = 0$) the field $\Phi(\vec{r}, t)$ is proportional to the polar angle φ ($x = r\cos\varphi$, $y = r\sin\varphi$) and has the form

$$\Phi = \arctan\left(\frac{y}{x}\right). \tag{7}$$

The field $m(\vec{r})$ 'regularizes' the vortex at the core so that

$$m(0) = p \qquad m(r) \to 0 \qquad r \to \infty$$

where $p = \pm 1$ is the so-called polarity of the vortex. The polarity is a constant of motion (and thus a second topological charge) only in the continuum limit. In the discrete spin model p can change during the movement and can flip between its two extreme values ± 1 under the action of thermal noise [13] or a rotating in-plane magnetic field [14]. The magnetization field is easier to study by rescaling distances $\rho = r\sqrt{K}$ and introducing the new field θ by $m = f(\rho) = \cos(\theta(\rho))$, which satisfies the equation

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}2\theta}{\mathrm{d}\rho}\right) + \left(1 - \frac{1}{\rho^2}\right)\sin(2\theta) = 0. \tag{8}$$

The θ field has the following asymptotic behaviour $\theta = a\rho(1 - \rho^2/8)$ for $\rho \to 0$ and $\theta = \pi/2 - b\sqrt{\frac{\pi}{2\rho}}\,\mathrm{e}^{-\rho}/2$ for $\rho \to \infty$ so that we obtain for the magnetization m

$$f = 1 - a\rho^2 \qquad \rho \to 0 \qquad f = b\frac{\mathrm{e}^{-\rho}}{\sqrt{\rho}} \qquad \rho \to \infty. \tag{9}$$

To describe a vortex pair it is then natural to follow a superposition principle for ϕ and assume the following ansatz for the fields:

$$m(\vec{r},t) = \sum_{i=1,2} m_i \equiv \sum_{i=1,2} f(\kappa_i(t)|\vec{r}-\vec{R}_i(t)|)$$
$$\phi(\vec{r},t) = \sum_{i=1,2} \phi_i \equiv \sum_{i=1,2} q_i \Phi(\vec{r}-\vec{R}_i(t)) \tag{10}$$

where $\vec{R}_i(t)$ is the position of the centre and $\kappa_i(t)$ the inverse width of the out-of-plane component of the ith vortex with the topological charge q_i ($q_1 = -q_2 = 1$). κ_i will depend in general on the anisotropy and below we will show how. We will also assume that the vortices are well separated from each other so that the m_i fields do not overlap.

Inserting equations (10) into equation (6) we obtain the energy of the vortex–antivortex pair up to a constant as (see the appendix)

$$W = 2\pi \ln(\sqrt{\kappa_1\kappa_2}|\vec{R}_1-\vec{R}_2|) + U(\kappa_1,\vec{R}_1) + U(\kappa_2,\vec{R}_2) \tag{11}$$

where the first term describes the vortex–antivortex interaction while

$$U(\kappa,\vec{R}) = \frac{1}{2}\int \mathrm{d}\vec{r}\, K(\vec{r}) f^2(\kappa|\vec{r}-\vec{R}|) \tag{12}$$

is the magnetic anisotropy energy in a single-vortex state. Note that we have assumed the vortices to be well separated and neglected the additional interaction term between them due to the spatial dependence of the anisotropy parameter $K(\vec{r})$. This is valid because the out-of-plane component m of the vortex decays exponentially [6].

From expression (12) one can build a collective variable approach for the dynamics of one vortex in the Thiele approach [6]. In the homogeneous case, inserting a one-vortex ansatz in the Hamiltonian (6) and integrating over the domain (assumed here finite) one obtains the Thiele equation for a single vortex as [9]

$$\vec{G}\times\dot{\vec{R}} = \frac{\partial}{\partial\vec{R}}U.$$

This can also be obtained from the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\vec{G}\cdot(\dot{\vec{R}}\times\vec{R}) - U(\vec{R})$$

where $\vec{G}$ is the gyrocoupling vector and U is the energy of the single vortex [14].

It is then natural to describe the dynamics of a vortex pair in the inhomogeneous case by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\sum_{i=1,2}\vec{G}_i\cdot(\dot{\vec{R}}_i\times\vec{R}_i) - W \tag{13}$$

and the associated Thiele equations

$$\vec{G}_i\times\dot{\vec{R}}_i = \frac{\partial}{\partial\vec{R}_i}W.$$

In the case under consideration, $\vec{G}_i = 2\pi q_i p_i \vec{e}_z$ where q_i is the topological charge of the ith vortex and p_i is its polarity. We will consider the case when $p_1 = p_2 = 1$.

Note that the Thiele approximation in its proper sense corresponds to the assumption that the shape of the vortex is rigid (see, however, [6, 12] where extensions of the Thiele approach were proposed). We relax this condition by considering the width of each vortex κ_i to be slaved to the position of the centre $\vec{R}_i$. This means that in the Lagrangian we omit the kinetic energy term for the width variables, assuming that their time dependence is determined by their associated vortex positions. From the Lagrangian (13) we obtain that the dynamics of the vortex–antivortex pair is governed by the following set of equations:

$$\vec{G}_i \times \dot{\vec{R}}_i = \frac{\partial}{\partial \vec{R}_i} W \tag{14}$$

$$\frac{\pi}{\kappa_i} + \frac{\partial}{\partial \kappa_i} U(\kappa_i, \vec{R}_i) = 0 \qquad i = 1, 2. \tag{15}$$

3. Vortex–antivortex propagation across an interface

We consider here a system with a one-dimensional interface separating two 2D media of anisotropy K_L on the left and K_R on the right, so that K can be written as

$$K(\vec{r}) \equiv K(x) = (K_R - K_L)H(x) + K_L \tag{16}$$

where $H(x)$ is the Heaviside function. We will consider the particular case of normal incidence of the vortex–antivortex pair with respect to the interface. It is seen from equations (14) and (15) that if $X_1 = X_2$ at initial time $t = t_0$, then this equality holds for all t and the dynamics of the vortex–antivortex pair is described by the equations

$$\dot{X} = \frac{1}{Y} \qquad \dot{Y} = -\frac{1}{\pi}\frac{\partial}{\partial X} U(\kappa, X) \tag{17}$$

$$\frac{\pi}{\kappa} + \frac{\partial}{\partial \kappa} U(\kappa, X) = 0 \tag{18}$$

where

$$X \equiv X_1 = X_2 \qquad Y = Y_1 - Y_2 \qquad \kappa \equiv \kappa_1 = \kappa_2.$$

Before analysing in detail the equations of motion it is useful to make some remarks using the energy of the system. Let us consider the two limiting cases where the vortex pair is well inside the medium to the left (respectively the right) of the interface so that $K = K_L$ (respectively $K = K_R$). In the first case we obtain

$$U(\kappa, X) = \frac{\pi}{\kappa^2} K_L$$

where we have taken into account (see e.g. [10])

$$2\pi \int_0^\infty \cos^2(\theta(r)) r \, \mathrm{d}r = \pi.$$

The evolution equation (18) for κ then yields

$$\kappa^2 = K_L.$$

Away from the interface the vortex separation is constant $Y = Y_L$ as given by (17). The total energy of the vortex pair can then be written as

$$W_L = 2\pi \log(\kappa_L Y_L) + \pi = 2\pi \log(\sqrt{K_L} Y_L) + \pi. \tag{19}$$

The same considerations done when the vortex pair is well inside the medium to the right of the interface give

$$W_R = 2\pi \log(\kappa_R Y_R) + \pi = 2\pi \log(\sqrt{K_R} Y_R) + \pi. \tag{20}$$

The system is Hamiltonian so that $W_L = W_R$ and therefore

$$Y_R = Y_L \sqrt{\frac{K_L}{K_R}}. \tag{21}$$

Note that this result holds independently of the profile $K(x)$ as long as $K(x) = K_L$ (respectively $K(x) = K_R$) for $x \to -\infty$ (respectively $x \to +\infty$).

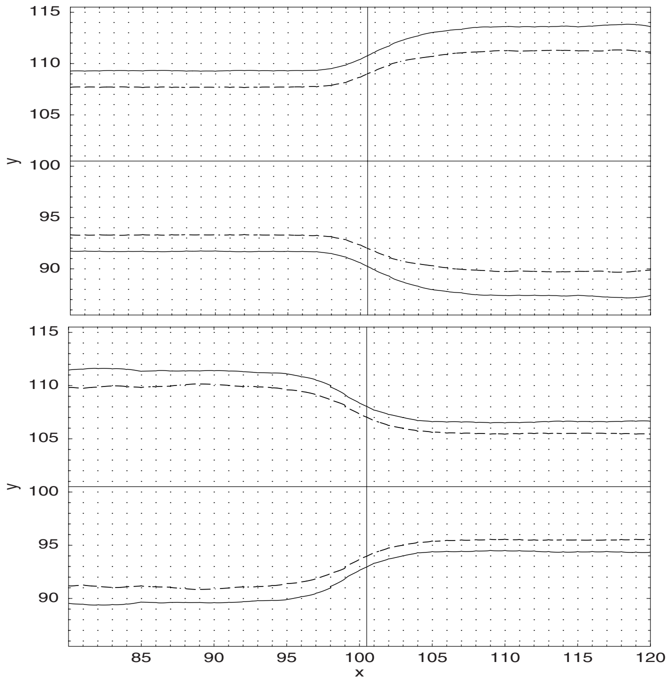

Figure 1. Trajectories of the centres of two vortex ($q = 1$, upper lines)–antivortex ($q = -1$, lower lines) pairs (one pair is in dashed lines, the larger pair is in solid lines), both vortices of a pair with positive polarization ($p = 1$), while crossing an interface (the vertical line at the centre of the system, whose diameter is $L = 200$). In the top panel the anisotropy changes from $K_L = 0.16$ on the left to $K_R = 0.06$ on the right. These settings are reversed in the bottom panel. The grid of points represents the lattice sites. The trajectories shown consist of points separated by a time interval $\delta t = 0.8$, which does not allow them to be resolved as separated points.

4. Numerical results

We have numerically integrated the full set of Landau–Lifshitz equations (2) over square lattices of sizes $N^2 = 120^2$ and 200^2 using a fourth-order Runge–Kutta scheme with time step 0.01. Each lattice is bounded by a circle of diameter $L = N$ on which the spins are free. This corresponds to Neumann boundary conditions in the continuum case and causes the presence of image vortices. We have made the domains so large in order to minimize their influence on the vortex pair. The anisotropy constant is $\mathcal{K}_L$ in the left half of the circle and $\mathcal{K}_R$ in the right half with a vertical interface separating the two media. We have fixed the exchange constant $J = 1$, the spin length $S = 1$, as well as the lattice constant $a = 1$, so that we can use the notation K_L and K_R for the anisotropy constants instead of $\mathcal{K}_L$ and $\mathcal{K}_R$.

To gain time the evaluation of the right-hand side and the time advance of the field were parallelized using OpenMP directives embedded in the Fortran code. A typical run of 10 000 steps takes about 1 h on four processors on a Silicon Graphics Origin 2000 machine with 64 RS10000 processors.

As the initial condition we used a vortex–antivortex pair moving horizontally towards the interface and tested several inter-vortex distances. This inter-vortex distance has to be on one hand much smaller than the size of the system in order to avoid the influence of images and on the other hand large enough to avoid overlapping of the out-of-plane structures of the two

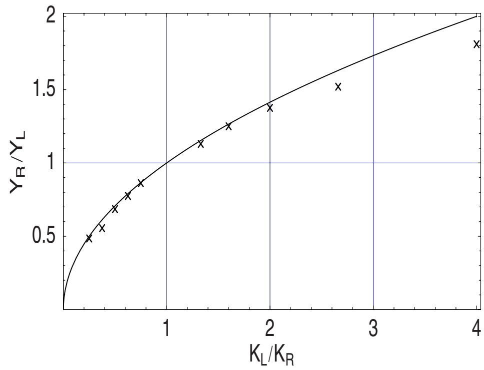

Figure 2. Plot of (Y_R/Y_L) as a function of K_L/K_R for different situations of vortex pairs impinging on an interface. The full curve shows the expression given by (21).

vortices before or after the pair has crossed the interface. We have studied mainly the system of diameter $L = 200$, for which the first condition was fulfilled and confirmed the second by examining the *m* field contour plots of the vortices. Before starting the integration a relaxation procedure was applied to adapt the vortex pair to the lattice. The position of the vortex centre is calculated using an interpolation. The details of the relaxation and the estimation of the centre can be found in the appendix of [12].

For most of the simulations, the values of anisotropy on both sides of the lattice were in the range $0.06 \leqslant K \leqslant 0.18$. In figure 1 typical events are shown, where the vortex–antivortex pair crosses an interface with $K_L = 0.16$ and $K_R = 0.06$.

Figure 2 shows $(Y_R/Y_L)\sqrt{(}K_L/K_R)$ as a function of K_L/K_R for different numerical experiments and gives a good agreement with the approximation (21). There is, however, a systematic bias so that Y_R is always overestimated. The reason for this is that at the sharp interface spin waves are generated which are not taken into account by the collective coordinate theory. In practice, this will always cause $W_L > W_R$ and the mismatch increases with the ratio K_L/K_R.

5. Conclusion

We have described quantitatively the influence of an interface on the distance between a vortex and an antivortex in a vortex pair. After the pair crosses the interface, the separation Y_R is given very simply by $Y_R = Y_L\sqrt{\frac{K_L}{K_R}}$, where Y_L is the initial separation when the pair is in the left medium and K_L (respectively K_R) is the anisotropy of the left (respectively right) medium. The model relies on a collective variable approximation where the vortex structure is assumed rigid and depends on time only through the positions R_i and core sizes κ_i of the individual components. In addition, we assume the core size to be slaved to the position R_i. Detailed comparisons were made using the numerical solutions of the Landau–Lifshitz equations obtained from (1), giving very good agreement with our simple expression.

A detailed study of the motion in order to predict the time evolution of the vortex centres is more involved because one needs to solve the differential algebraic system of equations (14), (15). We plan to do this in the near future.

The effect shown here for local anisotropy can also be observed for the case of exchange anisotropy where the second term of the Hamiltonian (1) is replaced by

$$+\frac{J}{2}\sum_{(\vec{n},\vec{a})}\delta_{\vec{n}}\left(\mathcal{S}^z_{\vec{n}}\cdot\mathcal{S}^z_{\vec{n}+\vec{a}}\right)$$

where $0<\delta_{\vec{n}}\ll 1$. The ratios of the distances Y_R/Y_L are again given by (21) with $\lambda=1-\delta$ in place of $\mathcal{K}$. This is to be expected from the fact that both systems have approximately the same continuum limit.

The effect described is very general and we expect to find the same type of phenomenon for vortices in other contexts, as long as the material properties that are changed at the interfaces only affect the vortex core as in the case described here.

Acknowledgments

The authors have benefitted from a French–German Procope grant (nb 04555TG). Part of the computations was done at the Centre de Ressources Informatiques de Haute-Normandie and JGC thanks Guy Moebs for help with the parallelization of the program. JGC is also grateful to Stavros Komineas for useful electronic conversations. JPZ is supported by Deutsche Forschungsgemeinschaft, at the University of Bayreuth, where part of the computations with the original code was performed.

Appendix. Energy for the vortex–antivortex pair

We call H_1, H_2 and H_3 the three terms of H and evaluate each of them using the ansatz (10). We assume that the vortex pair is well separated so that the two out-of-plane structures $m_{1,2}$ do not overlap. To simplify the notation we will use underscores for partial derivatives.

The first term is

$$H_1=\frac{1}{2}\int \mathrm{d}\vec{r}\,\frac{(\nabla m)^2}{1-m^2}\approx\sum_{i=1,2}\frac{1}{2}\int \mathrm{d}\vec{r}\,\frac{(\nabla m_i)^2}{1-m_i^2}=2\pi\int_0^\infty \mathrm{d}\rho\,\rho(\theta_\rho)^2.$$

The second term in the Hamiltonian involves the long-range fields ϕ and we calculate it first over a large but finite domain of size L,

$$H_2=\frac{1}{2}\int \mathrm{d}\vec{r}(1-m^2)(\nabla\phi)^2\approx\sum_{i=1,2}\frac{1}{2}\int \mathrm{d}\vec{r}\left(1-m_i^2\right)(\nabla\phi_i)^2+\int \mathrm{d}\vec{r}(\nabla\phi_1)(\nabla\phi_2).$$

Each term in the sum can be written as

$$\frac{1}{2}\int \mathrm{d}\vec{r}\big(1-m_i^2\big)(\nabla\phi_1)^2\approx\pi\log(\kappa_i L)+H_2^0$$

where

$$H_2^0=\int_0^\infty \mathrm{d}\rho\frac{\sin^2(\theta)}{\rho}-\int_1^\infty \mathrm{d}\rho\frac{\cos^2(\theta)}{\rho}.$$

To compute the last term, we take the first vortex as the origin and write $R=|R_1-R_2|$. We obtain

$$\int \mathrm{d}\vec{r}(\nabla\phi_1)(\nabla\phi_2)=q_1q_2\int_0^{2\pi}\mathrm{d}\psi\int_0^r \mathrm{d}r\,\frac{r^2-rR\cos(\psi)}{r^2+R^2-2rR\cos(\psi)}=2\pi q_1q_2\log\left(\frac{L}{R}\right).$$

Finally

$$H_2=2\pi q_1q_2\log\left(\frac{L}{|R_1-R_2|}\right)+\pi\log(\kappa_1L)+\pi\log(\kappa_2L)+2H_2^0$$

and we recover a finite energy for $L \to +\infty$ for a vortex–antivortex pair $q_1 q_2 = -1$

$$H_2 = 2\pi \log(\sqrt{\kappa_1 \kappa_2}|R_1 - R_2|)$$

where we omitted the constant term $2H_2^0$, a well-known result for the case $\kappa_1 = \kappa_2$ [11].

The last term can be approximated as

$$H_3 = \frac{1}{2}\int \mathrm{d}\vec{r}\, K(\vec{r}) m^2 \approx \sum_{i=1,2} \frac{1}{2}\int \mathrm{d}\vec{r}\, K(\vec{r}) m_i^2 = \sum_{i=1,2} U(\kappa_i, \vec{R}_i)$$

where

$$U(\kappa, \vec{R}) = \frac{1}{2\kappa^2}\int \mathrm{d}\vec{\rho}\, f^2(\rho) K\left(\frac{\vec{\rho} + \kappa \vec{R}}{\kappa}\right).$$

Combining H_1, H_2 and H_3 we obtain the reduced Hamiltonian (11).

References

[1] Lamb H 1990 *Hydrodynamics* (New York: Dover)
[2] Saffman P G 1992 *Vortex Dynamics* (Cambridge: Cambridge University Press)
[3] Vasnetsov V and Staliunas K (ed) 1999 *Optical Vortices* (Commack, NY: Nova)
[4] Stratopoulos G N and Tomaras T N 1996 *Phys. Rev.* B **54** 12493
[5] Mertens F G and Bishop A R 2000 *Nonlinear Science at the Dawn of the 21st Century* ed P L Christiansen, M P Soerensen and A C Scott *(Lecture Notes in Physics)* (Berlin: Springer)
[6] Vökel A R, Wysin G M, Mertens F G, Bishop A R and Schnitzer H J 1994 *Phys. Rev.* B **50** 12711
[7] Lavrentiev M and Shabat B V 1972 *Problemes hydrodynamiques et variables complexes* (Moscow: Mir)
[8] Thiele A A 1973 *Phys. Rev. Lett.* **30** 230
Thiele A A 1974 *J. Appl. Phys.* **45** 377
[9] Huber D L 1982 *Phys. Rev.* B **26** 3578
[10] Komineas S and Papanicolaou N 1998 *Nonlinearity* **11** 265
[11] Kosterlitz J M and Thouless D J 1973 *J. Phys. C: Solid State* **6** 1181
[12] Mertens F G, Schnitzer H J and Bishop A R 1997 *Phys. Rev.* B **56** 2510
[13] Gaididei Yu, Kampetter T, Mertens F and Bishop A R 1999 *Phys. Rev.* B **59** 7010
[14] Zagorodny J P, Gaididei Yu, Mertens F G and Bishop A R 2003 *Eur. Phys. J.* B **31** 471
Gaididei Yu, Kampetter T, Mertens F and Bishop A R 2000 *Phys. Rev.* B **61** 9449